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A study on the multiple reciprocity method and complex-valued formulation for the Helmholtz equation

W. Yeih, J. T. Chen*, K. H. Chen & F. C. Wong

Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan 202, R.O.C.

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The relation between the multiple reciprocity method and the complex-valued formulation for the Helmholtz equation is re-examined in this paper. Both the singular and hypersingular integral equations derived from the conventional multiple reciprocity method are identical to the real parts of the complex-valued singular and hypersingular integral equations, provided that the fundamental solution chosen in the multiple reciprocity method is proper. The problem of spurious eigenvalues occurs when we use either a singular or hypersingular equation only in the multiple reciprocity method because information contributed by the imaginary part of the complex-valued formulation is lost. To filter out the spurious eigenvalues in the conventional multiple reciprocity method, singular and hypersingular equations are combined together to provide sufficient constraint equations. Several one-dimensional examples are used to examine the relation between the conventional multiple reciprocity method and the complex-valued formulation. Also, a new complete multiple reciprocity method in one-dimensional cases, which involves real and imaginary parts, is proposed by introducing the imaginary part in the undetermined coefficient in the zeroth-order fundamental solution. Based on this complete multiple reciprocity method, it is shown that the kernels derived from the multiple reciprocity method are exactly the same as those obtained in the complex-valued formulation.

1 INTRODUCTION

To study the eigenvalue problem of the scalar-valued Helmholtz equation in a bounded area, boundary element methods that are based on the integral equation have been employed. Comparing with the finite element method, the boundary element method seems to be very efficient since it requires discretization on the boundary only. When the fundamental solution of the Helmholtz equation is used, the unknown wave number k is included in the fundamental solution.[1,2] This is the reason why the boundary element method is, sometimes, thought to be unsuitable for eigenvalue analysis despite its advantage in sole boundary discretization. Use of this kind of integral equation to deal with eigenvalue problems has been studied by many researchers.[3–6]

Another category is the integral equation formulation, which employs fundamental solution of the Laplace operator. However, this kind of integral equation inevitably produces an additional domain integral. To compute the domain integral, three methods are available:

1. the internal cell method;
2. the dual reciprocity method (DRM);
3. the multiple reciprocity method (MRM).

The first method requires discretization of the domain and, therefore, is less efficient compared with the others. Also, this method loses the spirit of the boundary element method, which uses discretization on the boundary only. The DRM is devised to transform the domain integral into the corresponding boundary integral with the help of additional application of Green's identity.[7] This method requires a special influence type interpolation function to approximate the unknown function inside the domain and some internal points besides the boundary nodes. The resulting formulation is domain discretization free.[8,9] MRM uses a

*Author to whom all correspondence should be addressed.

series of higher-order fundamental solutions derived from the Laplace operator (real-valued higher order fundamental solutions). This formulation is convenient during recomputation at different values of k because k is left outside the integrals. Many publications have focused on use of MRM to deal with eigenvalue problems of the Helmholtz equation.[10–12] For solving the diffusion problem, the complex-valued fundamental solutions derived from the Helmholtz operator has been used.[13] Without misleading the readers, the term 'MRM' cited in this paper only means the method based on the Laplace-type fundamental solutions which is intended for solving the Helmholtz equation.

Chen and Wong[14] found that MRM encounters the spurious eigenvalue problem in the one-dimensional case, provided that either a singular or hypersingular equation for MRM is used only, and they proposed a method for combining both singular and hypersingular equations for MRM to solve such a problem. The same problem appears in the two-dimensional case found by Wong.[15] However, the mechanism for spurious eigenvalues in the MRM formulation is not explained in their work.[14,15] Kamiya *et al.*[16] derived the complex-valued formulation for the scalar-valued Helmholtz equation, and they found that the MRM formulation only obtains one part of their complex-valued formulation in two-dimensional cases. They did not provide a general proof of their statement for any dimensional cases. The principal objectives of this paper are:

1. to obtain a general derivation of the relation between the complex-valued formulation and MRM for one-dimensional, two-dimensional and three-dimensional cases;
2. to give the reason for the occurrence of spurious eigenvalues in MRM and to explain how to deal with this kind of problem;
3. to propose a complete MRM that is fully equivalent to the complex-valued formulation.

2 FORMULATIONS AND RELATION BETWEEN THE CONVENTIONAL MRM AND COMPLEX-VALUED FORMULATION

The problem considered here is a scalar-valued Helmholtz differential equation of the potential u in a domain Ω bounded by the boundary Γ:

$$\nabla_x^2 \mathrm{u}(x) + k^2 \mathrm{u}(x) = 0 \text{ (for } x \text{ in } \Omega), \tag{1}$$

where ∇_x^2 is the Laplacian operator with respect to point x, k is the wave number, which is unknown for the eigenvalue problem, and $\mathrm{u}(x)$ is the unknown eigenmode. Eqn (1) is reduced to the singular integral equation as

$$c\mathrm{u}(\xi) + \int_\Gamma \mathrm{T}(x,\xi)\mathrm{u}(x)d\Gamma(x) - \int_\Gamma \mathrm{U}(x,\xi)\mathrm{t}(x)d\Gamma(x) = 0, \tag{2}$$

where $\mathrm{t}(x) \equiv \frac{\partial \mathrm{u}(x)}{\partial n_x}$ is the normal derivative of the unknown potential with n_x representing the outnormal direction at x point on the boundary, $\mathrm{U}(x,\xi)$ is the fundamental solution that satisfies both $(\nabla_x^2 + k^2)\mathrm{U}(x,\xi) = \delta(x-\xi)$ and the radiation condition, $\mathrm{T}(x,\xi) \equiv \frac{\partial \mathrm{U}(x,\xi)}{\partial n_x}$, and the value of c depends on where ξ is located.

Taking the normal derivative of eqn (2) with respect to n_ξ, the hypersingular equation can be obtained as

$$c\mathrm{t}(\xi) + \int_\Gamma \mathrm{M}(x,\xi)\mathrm{u}(x)d\Gamma(x) - \int_\Gamma \mathrm{L}(x,\xi)\mathrm{t}(x)d\Gamma(x) = 0 \tag{3}$$

where $\mathrm{L}(x,\xi) \equiv \frac{\partial \mathrm{U}(x,\xi)}{\partial n_\xi}$ and $\mathrm{M}(x,\xi) \equiv \frac{\partial^2 \mathrm{U}(x,\xi)}{\partial n_x \partial n_\xi}$. Eqns (2) and (3) together were called the dual integral equations by Chen and Hong.[17]

Another set of dual integral equations for an interior problem based on MRM can be also derived as:[14]

$$c\mathrm{u}(\xi) + \int_\Gamma \mathrm{T}_\mathrm{L}^\infty(x,\xi)\mathrm{u}(x)d\Gamma(x) - \int_\Gamma \mathrm{U}_\mathrm{L}^\infty(x,\xi)\mathrm{t}(x)d\Gamma(x) = 0 \tag{4}$$

$$c\mathrm{t}(\xi) + \int_\Gamma \mathrm{M}_\mathrm{L}^\infty(x,\xi)\mathrm{u}(x)d\Gamma(x) - \int_\Gamma \mathrm{L}_\mathrm{L}^\infty(x,\xi)\mathrm{t}(x)d\Gamma(x) = 0 \tag{5}$$

In the above equations, $\mathrm{U}_\mathrm{L}^\infty(x,\xi) = \sum_{j=0}^\infty (-k^2)^j \mathrm{U}_\mathrm{L}^{(j)}(x,\xi)$, where $\nabla_x^2 \mathrm{U}_\mathrm{L}^{j+1}(x,\xi) = \mathrm{U}_\mathrm{L}^{(j)}(x,\xi)$ and $\nabla_x^2 \mathrm{U}_\mathrm{L}^{(0)} = \delta(x-\xi)$, $T_\mathrm{L}^\infty \equiv \frac{\partial \mathrm{U}_\mathrm{L}^\infty}{\partial n_x}$, $\mathrm{L}_\mathrm{L}^\infty \equiv \frac{\partial \mathrm{U}_\mathrm{L}^\infty}{\partial n_\xi}$ and $\mathrm{M}_\mathrm{L}^\infty \equiv \frac{\partial^2 \mathrm{U}_\mathrm{L}^\infty}{\partial n_x \partial n_\xi}$. These kernel functions are all real functions since they are derived from the Laplace differential operator. The subscript 'L' in these kernels denotes that they are all derived from the Laplace type operator and the superscript '∞' means all the kernels are obtained from summing the infinite series.

For the eigenvalue problem of an interior domain, it is of great interest to determine whether these two formulations (one based on the complex-valued formulation and the other based on the real-valued MRM formulation) can yield the same result. To answer this question, we will first look at the relationship between them. Kamiya *et al.*[16] found by providing a two-dimensional example that the MRM formulation is identical to the real part of the complex-valued formulation. Their finding is conditionally true for general cases and will be discussed in the following.

Define that $\mathrm{U}^*(x,\xi) \equiv \mathrm{U}_\mathrm{L}^\infty(x,\xi) - \mathrm{U}(x,\xi)$, one finds

$$\nabla_x^2 \mathrm{U}^*(x,\xi) = \nabla_x^2 \mathrm{U}_\mathrm{L}^\infty(x,\xi) - \nabla_x^2 \mathrm{U}(x,\xi). \tag{6}$$

Recalling that

$$\nabla_x^2 \mathrm{U}(x,\xi) = \delta(x-\xi) - k^2 \mathrm{U}(x,\xi) \tag{7}$$

and

$$\nabla_x^2 \mathrm{U}_\mathrm{L}^\infty(x,\xi) = \sum_{j=0}^\infty (-k^2)^j \nabla_x^2 \mathrm{U}_\mathrm{L}^{(j)}(x,\xi) \tag{8}$$

$$= \sum_{j=1}^\infty (-k^2)^j \nabla_x^2 \mathrm{U}_\mathrm{L}^{(j)}(x,\xi) + \nabla_x^2 \mathrm{U}_\mathrm{L}^{(0)}(x,\xi)$$

$$= \sum_{j=1}^\infty (-k^2)^j \mathrm{U}_\mathrm{L}^{(j-1)}(x,\xi) + \delta(x-\xi),$$

substituting eqns (7) and (8) into eqn (6), one obtains

$$\nabla_x^2 U^*(x,\xi) = \sum_{j=1}^{\infty}(-k^2)^j U_L^{(j-1)}(x,\xi) + k^2 U(x,\xi) \quad (9)$$

$$= (-k^2)\sum_{j=0}^{\infty}(-k^2)^j U_L^{(j)}(x,\xi) + k^2 U(x,\xi)$$

$$= (-k^2)\left[U_L^{\infty}(x,\xi) - U(x,\xi)\right]$$

$$= -k^2 U^*(x,\xi).$$

Therefore, it can be concluded that

$$\nabla_x^2 U^*(x,\xi) + k^2 U^*(x,\xi) = 0. \quad (10)$$

This means that U^* can be any potential field satisfying the Helmholtz equation. Since the fundamental solutions U_L^{∞} and U are both functions of distance, it is required that

$$U^*(x,\xi) = U^*(r), \quad (11)$$

where r is defined as the distance between position x and ξ. Therefore, eqn (10) can be modified as

$$\nabla_r^2 U^*(r) + k^2 U^*(r) = 0 \quad (12)$$

where ∇_r^2 is the Laplacian operator in the radial part.

Kamiya *et al.*[16] claimed that the kernel derived from the real-valued MRM is identical to the real part of the complex-valued formulation. From the above discussion, it is clearly seen that the difference between the kernels of the single layer potential derived from the conventional MRM and the complex-valued formulation can be any potential field satisfying the Helmholtz equation, eqn (12). Therefore, Kamiya's finding is true when one specifies that $U^* = -i\text{Im}(U)$.

Furthermore, we should discuss whether the difference potential, U^*, should be embedded in U_L^{∞} or U. Assuming that U^* appears in U, $\bar{U} \equiv U - U^*$ is also a fundamental solution of the Helmholtz equation. However, although $\bar{U}$ can satisfy the governing eqn (1), it cannot satisfy the radiation condition at infinity. Therefore, U^* can only appear in U_L^{∞}. This means that $\bar{U}_L^{\infty} = U_L^{\infty} + U^*$ is also a representation of a kernel in the MRM formulation. Since the kernel in MRM is simply derived from the Laplace differential operator, there is no constraint boundary condition at infinity. Remember that U_L^{∞} was originally written in series form, in which every term is derived from the recursive formula; it would be interesting to know if U^* can be also derived from the recursive formula. The answer is yes, and this will be demonstrated in the next section using one dimensional cases. If one requires that the fundamental solution derived from MRM must satisfy the radiation condition, a new complete MRM formulation can be derived, which is fully equivalent to the complex-valued formulation. This will also be demonstrated by means of one-dimensional examples.

3 ONE-DIMENSIONAL EXAMPLES (CONVENTIONAL MRM, COMPLETE MRM AND COMPLEX-VALUED FORMULATION)

For a one-dimensional example, the domain is defined to be $0 \le x \le 1$ without loss of generality. Then, the governing

Table 1. Kernels of the conventional MRM and the complex-valued formulation for one-dimensional cases

	Conventional MRM	Complex-valued formulation
Governing equations	$\nabla^2 U_L^{(j+1)} = U_L^{(j)}\ j=1,2,\cdots$ $\nabla^2 U_L^{(0)} = \delta(r)$	$(\nabla^2 + k^2)U = \delta(r)$
Fundamental solutions	$U_L^{(j)} = \frac{1}{2}\frac{r^{2j+1}}{(2j+1)!}\ j=0,1,2,\ldots$	$U = \frac{e^{ikr}}{2ik}$
U kernel	$U_L^{\infty} = \sum_{j=0}^{\infty}\frac{(-k^2)^j}{2}\frac{r^{2j+1}}{(2j+1)!}$	$U = \frac{e^{ikr}}{2ik}$
T kernel	$T_L^{\infty} = \sum_{j=0}^{\infty}\frac{(-k^2)^j}{2}\frac{r^{2j}}{(2j)!}$ for $x > \xi$	$T = \frac{\cos(kr)}{2} + \frac{\sin(kr)}{2i}$ for $x > \xi$
	$T_L^{\infty} = -\sum_{j=0}^{\infty}\frac{(-k^2)^j}{2}\frac{r^{2j}}{(2j)!}$ for $x < \xi$	$T = \frac{-\cos(kr)}{2} + \frac{\sin(kr)}{2i}$ for $x < \xi$
L kernel	$L_L^{\infty} = -\sum_{j=0}^{\infty}\frac{(-k^2)^j}{2}\frac{r^{2j}}{(2j)!}$ for $x > \xi$	$L = \frac{-\cos(kr)}{2} + \frac{\sin(kr)}{2i}$ for $x > \xi$
	$L_L^{\infty} = \sum_{j=0}^{\infty}\frac{(-k^2)^j}{2}\frac{r^{2j}}{(2j)!}$ for $x < \xi$	$L = \frac{\cos(kr)}{2} + \frac{\sin(kr)}{2i}$ for $x < \xi$
M kernel	$M_L^{\infty} = -\sum_{j=0}^{\infty}\frac{(-k^2)^{j+1}}{2}\frac{r^{2j+1}}{(2j+1)!}$	$M = \frac{k\sin(kr)}{2} + \frac{k\cos(kr)}{2i}$ for $x > \xi$
		$M = \frac{k\sin(kr)}{2} - \frac{k\cos(kr)}{2i}$ for $x < \xi$

equation for this one-dimensional eigenproblem is formulated as

$$\frac{d^2u(x)}{dx^2}+k^2u(x)=0 \quad (0 \le x \le 1). \tag{13}$$

The kernel functions used in the conventional MRM and the complex-valued formulation are given in Table 1.[14,18] It is easily found that the real parts of the kernels derived using both methods are exactly the same since the zeroth order fundamental solution used in MRM is $r/2$.

Three examples with different boundary conditions are selected as the benchmark problems[14] and are listed as follows:

1. Case 1: u(0) = u(1) = 0 (Dirichlet problem);
2. Case 2: t(0) = t(1) = 0 (Neumann problem);
3. Case 3: u(0) = t(1) = 0 (Mixed problem).

To calculate the eigenvalues, several techniques are available.[14,16] The results of eigenvalue analysis for the first three modes using the conventional MRM and the complex-valued formulation are shown in Table 2. It can be seen in Table 2 that the same results are obtained for case (1) and case (2) using the conventional MRM and the complex-valued formulation. However, the conventional MRM encounters the spurious eigenvalue problem for case (3) when either the singular or hypersingular formulation is used only. On the other hand, the complex-valued formulation has no difficulty. Examining eqns (2) and (4) for the singular formulation (or eqns (3) and (5) for the hypersingular formulation), the only difference between them is that the complex-valued formulation has the imaginary part in the kernel, but the MRM formulation does not (see Table 1). Therefore, an eigenmode which satisfies eqn (2) (or eqn (3)) is, of course, the solution of eqn (4) (or eqn (5)). For the reverse statement, a solution of eqn (4) (or eqn (5)) is not necessarily a solution of eqn (2) (or eqn (3)). This means that a solution of eqn (4) (or eqn (5)) is not necessarily the real eigenmode. Since the conventional MRM only represents the real part in the complex-valued formulation, it loses information contributed by the imaginary part in the complex-valued formulation. Consequently, the spurious eigenvalue phenomenon appears in the conventional MRM formulation. It has been reported that the spurious eigenvalues are caused by the lack of imaginary part of the boundary integral equation.[19,20]

Since the spurious eigenvalue problem results from losing the imaginary part in MRM, it is natural to seek another constraint condition. Chen and Wong[14] proposed a technique for combining the singular and hypersingular formulations (eqn (4) and eqn (5)) together in the MRM to filter out the spurious eigenvalue by examining the eigenmode corresponding to the spurious eigenvalue. From Table 2, it can be found that their method can filter out the spurious eigenvalues. Based on the results of our research and the work by Tai and Shaw,[19,20] their idea can be implemented to find another constraint equation which can reconstruct information lost in the imaginary part of the complex-valued formulation.

The spurious eigenvalue problem in MRM can be solved in another way by constructing the complex-valued series form in MRM. From the above section, we know that the difference function between the kernels derived from MRM and from the complex-valued formulation can be any potential field satisfying eqn (12). This means that the zeroth

Table 2. Eigenvalue analysis by means of the conventional MRM and the complex-valued formulation

	Case (1)	Case (2)	Case(3)
Conventional MRM UT equation used only	3.14	3.14	1.57
	6.28	6.28	3.14*
	9.42	9.42	4.71
			6.28*
			7.85
			9.42*
Conventional MRM LM equation used only	3.14	3.14	1.57
	6.28	6.28	3.14*
	9.42	9.42	4.71
			6.28*
			7.85
			9.42*
Conventional MRM UT equation combined with LM equation	3.14	3.14	1.57
	6.28	6.28	4.71
	9.42	9.42	7.85
A complete MRM (Complex-valued formulation)	3.14	3.14	1.57
	6.28	6.28	4.71
	9.42	9.42	7.85

*The spurious eigenvalues.

order fundamental solution can be another function other than $r/2$. Choose the zeroth fundamental solution as

$$\bar{U}_L^{(0)}(r) = r/2 + U_L^{*(0)}(r). \quad (14)$$

This new zeroth order fundamental solution needs to satisfy $\nabla_r^2 \bar{U}_L^{(0)} = \delta(r)$, which means $\nabla_r^2 U_L^{*(0)} = 0$. Two homogeneous solutions can satisfy the above equation: r and 1. However, $r = 0$ may be encountered in the formulation, and r cannot be chosen since it produces a Dirac delta function at $r = 0$. Therefore, only a constant can be selected. This means that

$$\bar{U}_L^{(0)}(r) = r/2 + c, \quad (15)$$

where c is an arbitrary constant.

For higher order fundamental solutions, we can apply the recursive formula in the above section. Finally, we obtain

$$\bar{U}_L^{\infty}(r) = \frac{\sin(kr)}{2k} + c\cdot\cos(kr). \quad (16)$$

Comparing the kernel in eqn (16) to the kernel in the complex-valued formulation, we find that they are exactly the same if we choose $c = \frac{1}{2ik}$. One interesting thing should be pointed out here: this assignment makes the kernel of the single layer potential derived from MRM satisfy the radiation condition. Physically speaking, the conventional MRM which has only the real part kernels cannot be true for the eigenproblem since the phase angle behavior only appears in the imaginary part. Therefore, the conventional MRM is an incomplete formulation for the eigenproblem. On the other hand, the new derivation of MRM presented in this paper is a complete formulation. Since the kernel in this new formulation is exactly the same as that in the complex-valued formulation, the spurious eigenvalue will not occur here.

In the complete MRM formulation, we can derive the formulation by finding the higher-order fundamental solutions step by step as shown in this paper. However, the final series form of the kernels in MRM simply converge to corresponding kernels in the complex-valued formulation. For the eigenproblem, it is much easier to obtain the series forms of the kernels in MRM simply by expanding the kernels of the complex-valued formulation in a series.

Following the same method as shown for one-dimensional cases, the complete MRM formulation can be obtained by superposing an imaginary constant on the original zeroth order fundamental solution in the conventional MRM in two and three dimensional cases. The results are summarized in Table 3. In two dimensional

Table 3. The complete MRM and complex-valued formulations

Dimension	The zeroth-order fundamental solution used in the conventional MRM	The zeroth-order fundamental solution used in the complete MRM	The higher-order fundamental solution used in the complete MRM ($j \geq 1$)	U kernel in the complex-valued formulation
One-dimensional case	$\frac{r}{2}$	$\frac{r}{2} + \frac{1}{2ik}$	$\frac{1}{2}\frac{r^{2j+1}}{(2j+1)!} + \frac{1}{2ik}\frac{r^{2j}}{(2j)!}$	$\frac{e^{ikr}}{2ik}$
Two-dimensional case*	$\frac{1}{2\pi}\ln(r)$	$\frac{1}{2\pi}\ln(r) + \frac{1}{2\pi}\left(\gamma + \ln\frac{k}{2}\right) - \frac{1}{4i}$ with $\gamma = \lim_{j\to\infty}\left(\sum_{l=1}^{j}\frac{1}{l} - \ln j\right)$	$\frac{1}{4}\left\{\frac{2}{\pi}F_j(\ln r - S_j) + \frac{2}{\pi}F_j(\gamma + \ln\frac{k}{2}) + iF_j\right\}$ with $F_j = \frac{r^{2j}}{(j!)^2 4^j}$, $S_j = \sum_{l=1}^{j}\frac{1}{l}$	$\frac{i}{4}(J_0(kr) - iY_0(kr))$
Three-dimensional case	$\frac{-1}{4\pi r}$	$\frac{-1}{4\pi r} - \frac{1}{4\pi i}$	$\frac{-1}{4\pi}\left[\frac{r^{2j-1}}{(2j)!} - i\frac{r^{2j}}{(2j+1)!}\right]$	$\frac{-e^{-ikr}}{4\pi r}$

*The real part of the zeroth-order fundamental solution used in the conventional MRM and the complete MRM is different by a constant $\frac{1}{2\pi}\left(\gamma + \ln\frac{k}{2}\right)$.

cases, the zeroth-order fundamental solution of the Laplace equation is usually set to be $\frac{1}{2\pi}\ln r$. However, for constructing the complete MRM in which the real part of the single layer potential is equivalent to the real part of complex-valued formulation, the zeroth-order fundamental solution should be chosen in the way stated in Table 3.

4 CONCLUSIONS

MRM has been proved to be identical to one part of the complex-valued formulation for one, two and three dimensional cases, provided that the zeroth-order fundamental solutions used in MRM are chosen properly. Following this treatment, it is concluded that one can find the series representation of the kernels used in MRM by simply expanding the real part of the fundamental solution of the corresponding eigenvalue problem into the series form. The spurious eigenvalue phenomenon encountered in MRM using the singular formulation or the hypersingular formulation only has been explained. To deal with the spurious eigenvalue problem, another constraint equation is required. It has been suggested that either the complex-valued formulation or a combination of the singular and hypersingular equations in MRM can avoid the spurious eigenvalue problem. A new complete MRM formulation has also been proposed, which is fully equivalent to the complex-valued formulation.

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