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Relationship between the Green's matrix of SVD and the Green's function matrix of SVE for exterior acoustics

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摘要

本文主要目的在探討奇異值分解法在外域聲場中的物理意義。我們採用退化核函數及映射法推導得到格林函數，而這個格林函數同時可以奇異值展開式表示出。而格林函數離散後的矩陣可描述出一個輻射或散射物體的聲壓場與物體表面聲源強度的關聯。格林函數矩陣藉由奇異值分解法可得一組與輻射效率有關的奇異值，及兩組分別描述與場有關及與原點強度有關的正交的矩陣。此外，由奇異值分解法得到的酉向量與由奇異值展開式得到的基底函數之間的關聯將予以銜接。同時，藉由奇異值分解法得到的一圓形長柱體的輻射模態將與由奇異值展開式得到的解析解作一比較。

Abstract

In this paper, the principal objective is to study the physical meaning of the singular value decomposition (SVD) in exterior acoustics. The degenerate kernel and image method are employed to derive the Green's function. The Green's function can be represented by the singular value expansion (SVE). The Green's matrix describes the field of acoustic pressure to the strengths of sources on the surface of a body, which radiates or scatters sound. The matrix decomposed by the SVD technique resulted in a set of singular values and two sets of orthogonal singular vectors. The singular value relates to the radiation efficiency and the two sets of orthogonal unitary vectors describe field mode shapes and source mode shapes, respectively. In addition, the relationship between the unitary vectors provided by the SVD and the basis function provided by SVE is constructed. The acoustic radiation mode shape of a circular cylinder is obtained by using the SVD technique and is compared with the analytical solution by using the SVE.

Keywords：singular value decomposition (SVD)；singular value expansion (SVE)；radiation efficiency；radiation mode

I Introduction

Recently, the singular value decomposition (SVD) technique has been adopted to study the fictitious frequency [1,2] and the spurious eigenvalue [3,4] successfully. In analyzing the acoustic radiated power and radiation efficiency, the SVD technique also plays an important role. Chen [5] employed the eigenvalue analysis to examine the physical meaning of surface complex acoustic power and its relationship to acoustic radiation efficiency. Borgiotti [6] was the first to employ the SVD technique to analyze the radiation from a vibrating structure into the far field. Nelson and Kahana [7] used the SVD technique to decompose the Green's function. They tried to connect the decomposition and the basis functions of classical acoustics for three-dimensional case. It was found that the left and right singular vectors associated with the SVD related to the sampled spherical harmonics by a unitary transformation. However, the formulation of the transformation matrix is not clear in their paper. In the present work, we will focus on the relationship between the unitary vectors provided by the SVD and the basis function provided by the SVE. Based on the degenerate kernels, the image method is used to obtain the Green's function of the radiation field. A circular case is demonstrated to study the result of SVD and is compared to the result of the Green's function matrix. The Green's function matrix displayed in a singular value expansion (SVE) form. The relationship between the unitary vectors and the basis function will be connected.

II The image method of acoustic field

The Green's function, $G(x,s)$, relating to the acoustic pressure of field to the strengths of source on the boundary, satisfies

$$(\nabla^2 + k^2)G(x,s) = 2\pi\delta(x-s) \tag{1}$$

where $\delta(x-s)$ is the Dirac delta function. For the auxiliary system subject to the Neumann boundary condition, the Green's function must satisfy

$$\frac{\partial G}{\partial n_x} = 0, \quad (x \text{ on } B), \tag{2}$$

where n_x denotes the outnormal direction at the boundary point x. By employing the image method, we have

$$G(x,s) = U(x,s) + U(x,s') \tag{3}$$

where $U(x,s)$ is the fundamental solution and $U(x,s')$ is the fundamental solution of the image system with a point sink at the image point s'. By using the two bases of the first kind Hankel and Bessel functions of the n-th order and their derivatives, $H_n^{(1)}(kx)$, $J_n(kx)$, we can decompose the two-dimensional kernel function into

$$U(x,s) = \begin{cases} U^i(\rho,\bar{\phi};R,\theta) = \sum_{n=-\infty}^{\infty} \frac{-i\pi}{2} H_n^{(1)}(k\rho) J_n(kR) \Theta_n(\theta) \Theta_n^{+}(\bar{\phi}), & \rho > R, \\ U^e(\rho,\bar{\phi};R,\theta) = \sum_{n=-\infty}^{\infty} \frac{-i\pi}{2} H_n^{(1)}(kR) J_n(k\rho) \Theta_n(\theta) \Theta_n^{+}(\bar{\phi}), & \rho < R, \end{cases} \tag{4}$$

where "+" denote Hermintian conjugate and $\Theta_n(\bar{f}) = e^{in\bar{f}}$, $\Theta_n(q) = e^{inq}$, $x = (r, \bar{f})$, $s = (R, q)$ and $s' = (R', q)$ in the polar coordinate. The definitions of r, $\bar{f}$, R, q, R', r and r' are shown in Fig.1. We can rewrite the Green's function as follows:

$$G(x,s) = U^e(r, \bar{f}; R, q) - U^i(r, \bar{f}; R', q), \tag{5}$$

subject to the Neumann boundary condition

$$\frac{\partial G(x,s)}{\partial n_x} = \frac{\partial U^e(r, \bar{f}; R, q)}{\partial n_r} - \frac{\partial U^i(r, \bar{f}; R', q)}{\partial n_r} = 0. \tag{6}$$

When the field point x locates on the boundary of the circle with a radius a, substitution of Eq.(4) into Eq.(6), the relationship between the R' and R is obtained

$$J_n(kR') = \frac{H_n^{(1)}(kR) J_n'(ka)}{H_n'^{(1)}(ka)}. \tag{7}$$

By substituting Eqs.(4) and (7) into Eq.(5), and the symmetry property, the Green's function is derived,

$$G(s,x) = \frac{-ip}{2} \sum_{n=-\infty}^{\infty} \frac{H_n'^{(1)}(ka) J_n(kR) - H_n^{(1)}(kR) J_n'(ka)}{H_n'^{(1)}(ka)} H_n^{(1)}(kr) \Theta_n(\bar{f}) \Theta_n^+(q). \tag{8}$$

The acoustic pressure field $u(x)$ can be obtained

$$u(x) = icr_0 t(\hat{s}) \sum_{n=-\infty}^{\infty} \frac{H_n^{(1)}(kr)}{H_n'^{(1)}(ka)} \Theta_n(\bar{f}) \Theta_n^+(\hat{q}), \tag{9}$$

where c is the sound velocity and r_0 is the density and $t(\hat{s})$ is the velocity strength of a point source at $(a, \hat{q})$.

III The singular value decomposition for the Green's matrix

For the readers' convenience, the $[G]$ denotes the Green's matrix obtained by using BEM and the $[G(s,x)]$ denotes the Green's function matrix obtained by the Green's function in this chapter. In BEM implementation, the boundary of a circle is discretized into V constant elements. If the P field points and the V source points are considered and $\{u_B\}$ and $\{t_B\}$ denote the acoustic pressure and normal velocity vectors on the boundary, respectively, then the boundary and the domain integral equations can be modified and assembled by the following matrix form,

$$[T_B]\{u_B\} = [U_B]\{t_B\} \tag{10}$$

$$u(x) = [T_D]\{u_B\} - [U_D]\{t_B\} \tag{11}$$

where $\{u(x)\}$ is the vector whose elements define the field pressure for the domain point x, T_B, U_B are the boundary influence matrices on the boundary, T_D, U_D are the domain influence matrices, respectively. Substituting Eq.(10) into Eq.(11), we have

$$\{u(x)\} = ([T_D][T_B]^{-1}[U_B] - [U_D])\{t_B\} = [G]\{t_B\} \tag{12}$$

The SVD enables any arbitrary complex matrix [G] of order $P \times V$, the SVD of the Green's function matrix can be expressed in such that $[G] = \sum_{i=1}^{N} s_i f_i y_i^+$, where the Green's matrix is shown to consist of a linear superposition of N submatrices.

IV The singular value expansion of the Green's function matrix

We use Eq.(9) to define the elements of the Green's function $G(\hat{s}, x)$ relating the acoustic pressure at number of P points in the sound field to the source strength at number of V points on the boundary of the domain. The Green's function matrix can be written in the form

$$[G(\hat{s}, x)] = \lim_{M \to \infty} \begin{bmatrix} \sum_{n=-M}^{M} g_n \Theta_n(\bar{f}_1)\Theta_n^+(\hat{q}_1) & \cdots & \sum_{n=-M}^{M} g_n \Theta_n(\bar{f}_1)\Theta_n^+(\hat{q}_V) \\ \sum_{n=-M}^{M} g_n \Theta_n(\bar{f}_2)\Theta_n^+(\hat{q}_1) & \cdots & \sum_{n=-M}^{M} g_n \Theta_n(\bar{f}_2)\Theta_n^+(\hat{q}_V) \\ \vdots & \ddots & \vdots \\ \sum_{n=-M}^{M} g_n \Theta_n(\bar{f}_P)\Theta_n^+(\hat{q}_1) & \cdots & \sum_{n=-M}^{M} g_n \Theta_n(\bar{f}_P)\Theta_n^+(\hat{q}_V) \end{bmatrix}_{P \times V} \quad (13)$$

where $g_n = i r_0 c \dfrac{H_n^{(1)}(kr)}{H_n'^{(1)}(ka)}$. Since each term in the series comprising each element of the matrix is weighted by the same factor g_n, it is possible to write the matrix as a singular value expansion having the form $G(\hat{s}, x) = \lim_{M \to \infty} \sum_{n=-M}^{M} g_n \Omega_n(f_P)\Omega_n^+(q_V)$, in which $\Omega_n(f_P)$ and $\Omega_n(\hat{q}_V)$ are the left and right singular vectors, respectively.

V The singular value expansion and the singular value decomposition

It will be demonstrated by the numerical simulations presented below that there is indeed, under certain circumstances, a direct connection between the results of the components (Φ and Ψ) in the singular value decomposition for the Green's matrix and the matrices $\Theta(\bar{f}_P)$ and $\Theta(\hat{q}_V)$ of the SVE. Now, we connected the Φ, Ψ, $\Theta(\bar{f}_P)$ and $\Theta(\hat{q}_V)$ by a transformation matrix $\Gamma(\bar{f}_P)$ and $\Gamma(q_V)$, respectively.

The Green's matrix can be written as $[G] = \Theta(\bar{f}_P)\Gamma(\bar{f}_P)\Sigma_N \Gamma^+(\hat{q}_P)\Theta^+(\hat{q}_P)$, where Σ_N is the diagonal matrix of the N non-zero real singular values. It is evident from that the diagonal matrix Λ of the complex amplitudes is given by

$$\Lambda = \Gamma(\bar{f}_P)\Sigma_N \Gamma^+(q_V)\,. \quad (14)$$

We obtained the relationship between the Σ_N of the SVD and the Λ of the SVE.

VI Numerical examples

For the numerical experiments, we consider an infinite circular cylinder with radius $a = 1\,m$. Thirty points were adopted in the boundary element mesh for a circular boundary and observation field. The source points on the surface and the observation points are shown in Fig.2. The first five columns of Φ and Ψ matrices for the circular cylinder with thirty points at $r = 10\,m$, are shown in Fig.3 for the cases of *ka*=0.01. The dotted line and solid line denote the imaginary-part and the real-part of the vector, respectively. The *x* axis denotes the angular degrees of the position for the source points in y_i and for the observation points in f_i. The *y* axis denotes the amplitude of singular vectors in the Φ or Ψ matrices. Figure 3 show that the magnitude of the individual component is unchanged, but their phase may be different. The figure matches the harmonic bases in the SVD.

VII Conclusions

In this paper, we have demonstrated the effectiveness of the SVD technique in solving exterior acoustics. The physical meaning of the SVD has been examined. We applied the image method in conjunction with the degenerate kernel function to obtain the Green's function. The connection between the unitary vectors in the Green's matrix provided by the SVD and the function provided by the singular value expansion has been investigated. The unitary vectors are the basis functions for a diagonal transformation with respect to the generalized coordinate. The left and right singular vectors of the SVD of the Green's matrix yield two sets of orthogonal basis functions describing field mode shapes and source mode shapes, respectively.

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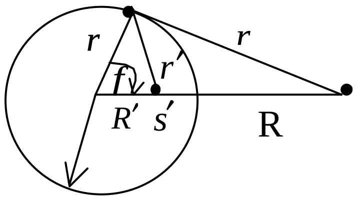

Fig.1 The definitions of

$\boldsymbol{r}, \boldsymbol{f}, R, \boldsymbol{q}$, R', r' and r.

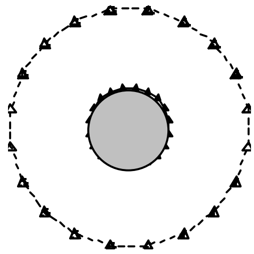

Fig.2 The nodes of boundary element mesh ▲ and observation points △

$\boldsymbol{f}$ $[G]_{30\times30}$ $\boldsymbol{j}$

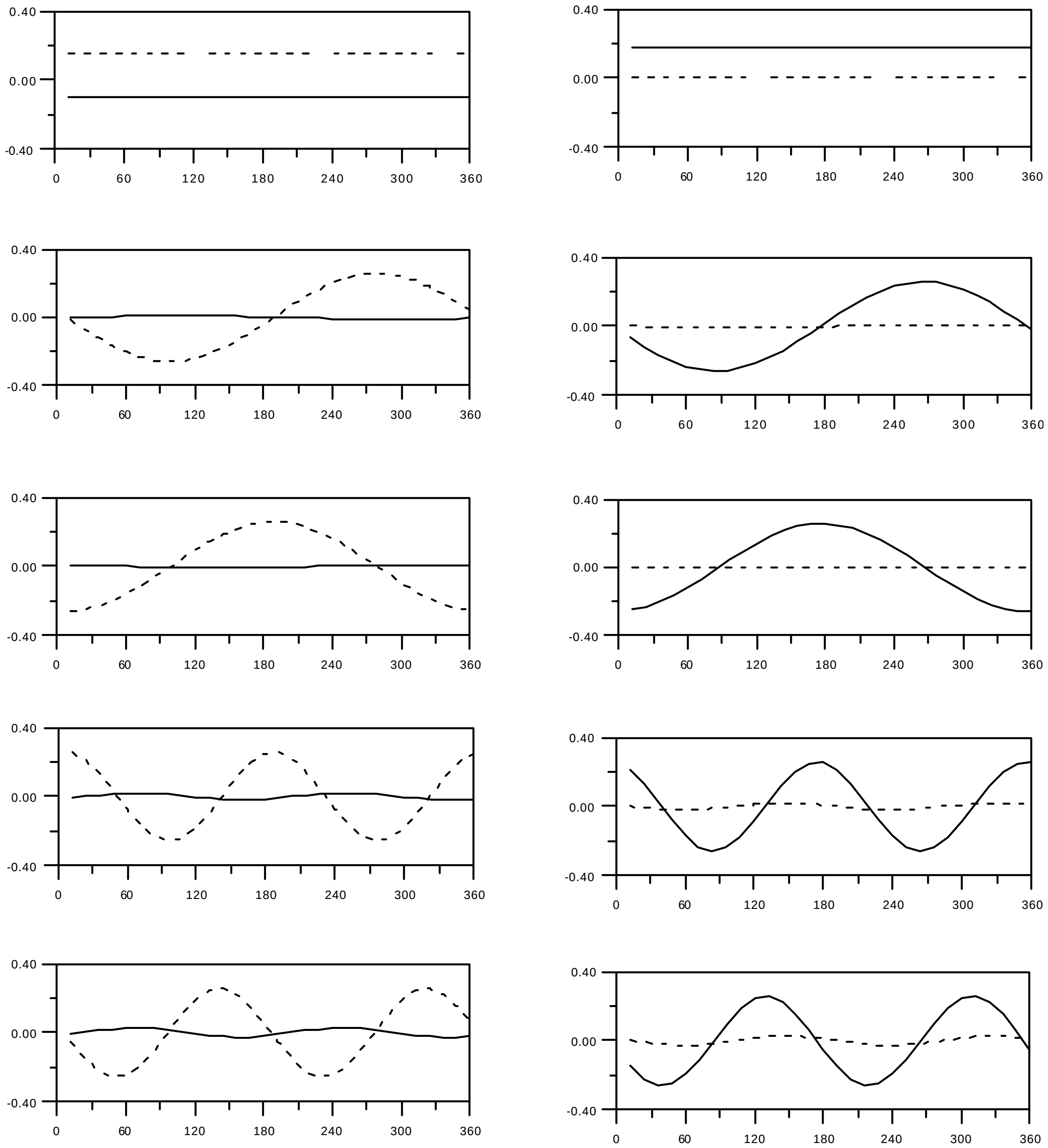

Figure 3. The first five columns of Φ and Ψ matrices for the circular cylinder at $\boldsymbol{r}$ =10.0 m for k=0.01 using thirty observation points.