LINEAR ALGEBRA
AND ITS APPLICATIONS

FIFTH EDITION

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in the sense that the sum of the squares of the *orthogonal* distances to the line is minimized. In fact, principal component analysis is equivalent to what is termed *orthogonal regression*, but that is a story for another day.

CHAPTER 7 SUPPLEMENTARY EXERCISES

1. Mark each statement True or False. Justify each answer. In each part, A represents an $n \times n$ matrix.

a. If A is orthogonally diagonalizable, then A is symmetric.

b. If A is an orthogonal matrix, then A is symmetric.

c. If A is an orthogonal matrix, then $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

d. The principal axes of a quadratic form $\mathbf{x}^TA\mathbf{x}$ can be the columns of any matrix P that diagonalizes A.

e. If P is an $n \times n$ matrix with orthogonal columns, then $P^T = P^{-1}$.

f. If every coefficient in a quadratic form is positive, then the quadratic form is positive definite.

g. If $\mathbf{x}^TA\mathbf{x} > 0$ for some $\mathbf{x}$, then the quadratic form $\mathbf{x}^TA\mathbf{x}$ is positive definite.

h. By a suitable change of variable, any quadratic form can be changed into one with no cross-product term.

i. The largest value of a quadratic form $\mathbf{x}^TA\mathbf{x}$, for $\|\mathbf{x}\| = 1$, is the largest entry on the diagonal of A.

j. The maximum value of a positive definite quadratic form $\mathbf{x}^TA\mathbf{x}$ is the greatest eigenvalue of A.

k. A positive definite quadratic form can be changed into a negative definite form by a suitable change of variable $\mathbf{x} = P\mathbf{u}$, for some orthogonal matrix P.

l. An indefinite quadratic form is one whose eigenvalues are not definite.

m. If P is an $n \times n$ orthogonal matrix, then the change of variable $\mathbf{x} = P\mathbf{u}$ transforms $\mathbf{x}^TA\mathbf{x}$ into a quadratic form whose matrix is $P^{-1}AP$.

n. If U is $m \times n$ with orthogonal columns, then $UU^T\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto Col U.

o. If B is $m \times n$ and $\mathbf{x}$ is a unit vector in $\mathbb{R}^n$, then $\|B\mathbf{x}\| \le \sigma_1$, where σ_1 is the first singular value of B.

p. A singular value decomposition of an $m \times n$ matrix B can be written as $B = P\Sigma Q$, where P is an $m \times m$ orthogonal matrix, Q is an $n \times n$ orthogonal matrix, and Σ is an $m \times n$ "diagonal" matrix.

q. If A is $n \times n$, then A and A^TA have the same singular values.

2. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for $\mathbb{R}^n$, and let $\lambda_1, \ldots, \lambda_n$ be any real scalars. Define

$$A = \lambda_1\mathbf{u}_1\mathbf{u}_1^T + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^T$$

a. Show that A is symmetric.

b. Show that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

3. Let A be an $n \times n$ symmetric matrix of rank r. Explain why the spectral decomposition of A represents A as the sum of r rank 1 matrices.

4. Let A be an $n \times n$ symmetric matrix.

a. Show that $(\text{Col } A)^\perp = \text{Nul } A$. [*Hint:* See Section 6.1.]

b. Show that each $\mathbf{y}$ in $\mathbb{R}^n$ can be written in the form $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, with $\hat{\mathbf{y}}$ in Col A and $\mathbf{z}$ in Nul A.

5. Show that if $\mathbf{v}$ is an eigenvector of an $n \times n$ matrix A and $\mathbf{v}$ corresponds to a nonzero eigenvalue of A, then $\mathbf{v}$ is in Col A. [*Hint:* Use the definition of an eigenvector.]

6. Let A be an $n \times n$ symmetric matrix. Use Exercise 5 and an eigenvector basis for $\mathbb{R}^n$ to give a second proof of the decomposition in Exercise 4(b).

7. Prove that an $n \times n$ matrix A is positive definite if and only if A admits a *Cholesky factorization*, namely, $A = R^TR$ for some invertible upper triangular matrix R whose diagonal entries are all positive. [*Hint:* Use a QR factorization and Exercise 26 in Section 7.2.]

8. Use Exercise 7 to show that if A is positive definite, then A has an LU factorization, $A = LU$, where U has positive pivots on its diagonal. (The converse is true, too.)

If A is $m \times n$, then the matrix $G = A^TA$ is called the *Gram matrix of A*. In this case, the entries of G are the inner products of the columns of A. (See Exercises 9 and 10.)

9. Show that the Gram matrix of any matrix A is positive semidefinite, with the same rank as A. (See the Exercises in Section 6.5.)

10. Show that if an $n \times n$ matrix G is positive semidefinite and has rank r, then G is the Gram matrix of some $r \times n$ matrix A. This is called a *rank-revealing factorization* of G. [*Hint:* Consider the spectral decomposition of G, and first write G as BB^T for an $n \times r$ matrix B.]

11. Prove that any $n \times n$ matrix A admits a *polar decomposition* of the form $A = PQ$, where P is an $n \times n$ positive semidefinite matrix with the same rank as A and where Q is an $n \times n$ orthogonal matrix. [*Hint:* Use a singular value decomposition, $A = U\Sigma V^T$, and observe that $A = (U\Sigma U^T)(UV^T)$.] This decomposition is used, for instance, in mechanical engineering to model the deformation of a material. The matrix P describes the stretching or compression of the material (in the directions of the eigenvectors of P), and Q describes the rotation of the material in space.