國立中山大學 應用數學 學系(研究所)

碩士論文

用基本解法求解 Laplace 方程的穩定性分析

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中華民國 九十五 年 六 月

前　　言

這篇論文共包含兩部分，第一部分是說明解決齊次方程邊界問題，基本解方法是先選擇滿足方程的基本解，運用線性結合的技巧去滿足內外邊界條件，為了避免對數奇異的問題，將 source points 放置在解的範圍之外上。基本解方法最早是在 1963 年由 Kupradze 提出，之後陸續有數值結果發表出來，但相關的分析卻很少。本文的第一部分是推導在 Neumann 與 Robin 邊界條件下的特徵值與估計當非圓形區域下邊界條件為混合的情況下其 condition number 的上界。在第一部分的最後則是數值上的測試，當問題為 Motz’s 問題時，利用基本解加上奇異方程式與基本解使用局佈加密的方法，都可以得到傳統的 condition number 會增加很大，但有效的 condition number 是適度的增大。然而，其係數在基本解方法中有很大的擾動得到不穩定，造成原因是因為減法消去的誤差再有限齊次解中造成的，因此在實際的應用上需同時考慮誤差與 ill-condition。

在論文的第二部分則是如何改善不穩定情況利用截斷奇異值分解法與 Tikhonov Regularization，這兩種方法其傳統的 condition number 跟有效的 condition number 計算公式與誤差分析都推導出，最後透過數值結果可以發現係數與 condition number 都有明顯的下降。

關鍵字：基本解方法、傳統的 condition number、有效的 condition number、截斷奇異值分解法、Tikhonov Regularization。

感　　言

時光匆匆，碩士班的生活即將結束，轉眼間求學的旅程也到了一個階段，在踏入社會之前，我要感謝一些在這階段督促我、陪伴我的人。

首先我要感謝我的家人，謝謝你們一路上支持著我，使我在生活上無後顧之憂，包容我的任性，即使在做錯了事時，也守在我身旁，更在迷惘、困惑的時刻，給了我一盞明燈，讓我有勇氣繼續往前走下去，你們是我最大的精神支柱。

感謝我的指導教授 — 李子才老師，謝謝你這兩年的指導，雖然我的表現並不出色，但老師並未因此而嫌棄我，一而再，再而三的教導我，從你身上我學到了“ 活到老 ”，學到老 ”的精神。

感謝這兩年在中山教導我的教授:呂宗澤老師、黃傑森老師與凌立雲老師，透過你們的授課與演講，讓我在知識學問上有進一步的成長，在疑或時求得答案。感謝口試委員:陳正宗教授、汪崧教授跟黃宏財學長，謝謝你們抽空參加我的論文口試，在論文上給我不同的意見。感謝系圖負責人 — 嘉鳳姊，謝謝你兩年的照顧，常常因為大小事情麻煩到你，真是辛苦你了。

最後謝謝兩年來陪伴在身邊的同學，2003-3 的伙伴們—鋒爺、峰哥、小叡、昭仁、賀僕、憲哥跟志鑫，你們讓研究室充滿歡笑，有你們在真好!瑋苓、士傑、洧立，還有女籃的伙伴們，感謝你們陪我渡過精采的時刻。

Stability Analysis of Method of Fundamental Solutions for Laplace's Equations

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June, 2006

Abstract

This thesis consists of two parts. In the first part, to solve the boundary value problems of homogeneous equations, the fundamental solutions (FS) satisfying the homogeneous equations are chosen, and their linear combination is forced to satisfy the exterior and the interior boundary conditions. To avoid the logarithmic singularity, the source points of FS are located outside of the solution domain S. This method is called the method of fundamental solutions (MFS). The MFS was first used in Kupradze [33] in 1963. Since then, there have appeared numerous reports of MFS for computation, but only a few for analysis. The part one of this thesis is to derive the eigenvalues for the Neumann and the Robin boundary conditions in the simple case, and to estimate the bounds of condition number for the mixed boundary conditions in some non-disk domains. The same exponential rates of Cond are obtained. And to report numerical results for two kinds of cases. (I) MFS for Motz's problem by adding singular functions. (II) MFS for Motz's problem by local refinements of collocation nodes. The values of traditional condition number are huge, and those of effective condition number are moderately large. However, the expansion coefficients obtained by MFS are oscillatingly large, to cause another kind of instability: subtraction cancellation errors in the final harmonic solutions. Hence, for practical applications, the errors and the ill-conditioning must be balanced each other. To mitigate the ill-conditioning, it is suggested that the number of FS should not be large, and the distance between the source circle and the ∂S should not be far, either.

In the second part, to reduce the severe instability of MFS, the truncated singular value decomposition(TSVD) and Tikhonov regularization(TR) are employed. The computational formulas of the condition number and the effective condition number are derived, and their analysis is explored in detail. Besides, the error analysis of TSVD and TR is also made. Moreover, the combination of TSVD and TR is proposed and called the truncated Tikhonov regularization in this thesis, to better remove some effects of infinitesimal σ_{min} and high frequency eigenvectors.

Keyword: method of fundamental solutions, traditional condition number, effective condition number, truncated singular value decomposition, Tikhonov regularization.

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Part I

Stability Analysis of Method of Fundamental Solutions

1 Introduction about Stability Analysis of MFS

In the first part of the thesis, to solve the boundary value problems of homogeneous equations, the fundamental solutions (FS) satisfying the homogeneous equations are chosen, and their linear combination is used to satisfy the exterior and the interior boundary conditions. To avoid the logarithmic singularity, the source points of FS are located outside of the solution domain S. This method is called the method of fundamental solutions (MFS). The MFS was first used in Kupradze [33] in 1963. Since then, there have appeared numerous reports of MFS for computation, see [9, 29, 30, 31, 42, 43], but only a few for analysis [4, 12, 31, 42]. Let us mention the important work here. By MFS, the numerical implements were provided in Mathon and Johnson [42], potential problems in Mey [43] and Katsurada and Okamoto [31], harmonic and bi-harmonic problems with singularities in Karageorghis [29], and eigenvalues of the Helmoholtz equation in Karageorghis [30]. Besides, the MFS by using the singular value decomposition (SVD) was developed in Ramachandran [46], and the truncated SVD (TSVD) in Chen et al. [8] to improve numerical stability. A systemic introduction on FMS is given in Chen et al. [9]. For the bounded domain S, when the source points of FS are located uniformly on an outside circle of S, the polynomial convergence rates were proved in Bogomolny [4], and the exponential convergence rates were provided in Katsurada and Okumoto [31]. The ill-conditioning (i.e., instability) is a severe issue of MFS. For Dirichlet problems, the exponential growth of the traditional condition number Cond was provided in Christiansen [12, 13] for the simple case that both the source and the collocation points are located uniformly on circles. However, it was pointed in Christansen [13], p. 385, that "for other problems than the two-dimensional Dirichlet problem it will be more difficult to determine the eigenvalues of the integral operator in closed form (perhaps impossible)". Extensive numerical experiments of Cond by MFS were reported in [12, 13]. The main goal of this thesis is to extend the stability analysis in [12, 13]. Some numerical results of both Dirichlet and Neumann boundary conditions by MFS are provided in [10, 11, 32], without strict proof of bounds for condition number. By the circulant matrix in Davis [17], the eigenvalues of the stiffness matrix from MFS can be obtained for the Dirichlet, the Neumann and the Robin conditions for the same case that the source and the collocation nodes are uniformly located on circles. When the stiffness matrix is singular, the solution methods can be provided by the singular value decomposition (SVD). Next, consider the mixed type of boundary conditions for some bounded and simply connected domains. We follow the approaches in [36], to obtain the bounds of Cond, where the inverse inequality is derived based on [4].

In this thesis, we will propose the effective condition number for stability analysis. The effective condition number was first used in Chan and Foulser [5], and then in Christiansen and Hansen [15] and Chirstiansen and Saranen [16]. Rencently, further exploitation of effective condition number is made in Li et al. [36], to give the new computational formula, Cond_eff. The values of Cond_eff may be smaller, or even much smaller than the traditional Cond. The analysis of Cond_eff is made for the finite difference method in [35], the collocation Trefftz method (TM) using particular solutions in [36], and the boundary integral equation in Huang et al. [28]. In this thesis, we apply Cond_eff for the MFS, which can also be regarded as the CTM using the fundamental solutions. The Trefftz methods [50] in 1926 are also called the boundary approximation method in Li [34], and the analysis of error and stability for CTM is given in [36, 37, 40]. In this thesis, the stability analysis of MFS is made, based on both Cond and Cond_eff.

Numerical experiments are carried out for two kinds of cases. (I) MFS for Motz's problem by adding singular functions. (II) MFS for Motz's problem. The values of Cond are huge, and those of Cond_eff are moderately large. However the expansion coefficients obtained by MFS are oscillatingly large, to cause another kind of instability: subtraction cancellation in the final harmonic solutions. Hence for practical applications, the errors and the ill-conditioning must be balanced each other. To mitigate the ill-conditioning, it is suggested that the number of FS should not be large, and the distance between the source circle and the ∂S should not be far, either.

This thesis is organized as follows. In the next section, the algorithms of MFS are described, and in Section 3, the effective condition number is provided for the least squares method (LSM) with rank deficient. In Section 4, the stability analysis of MFS is made for Dirichlet problems on disk domains, and in Section 5 for the Neumann and the Robin problems. In Section 6, the stability analysis is extend to the mixed type of boundary conditions for bounded and simply-connected domains, In Section 7, numerical experiments are carried out for Motz's problem by MFS, and in the least section, a few remarks are made.

2 Algorithms of Method of Fundamental Solutions

Consider Laplace's equation with the mixed type of the Dirichlet and the Robin boundary conditions,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad in \quad S, \tag{2.1}$$

$$u = f \quad on \quad \Gamma_D, \tag{2.2}$$

$$\frac{\partial u}{\partial \nu} + \alpha u = g \quad on \quad \Gamma_R, \tag{2.3}$$

where α is a non-negative constant, S is a bounded and simply-connected domain with the boundary $\partial S = \Gamma_D \cup \Gamma_R$, and ν is the exterior normal of Γ.

Denote in Figure 1,

$$r_{max} = \max_S r, \quad r_{min} = \min_S r. \tag{2.4}$$

A surrounding S is given by

$$\ell_R = \{(r,\theta) \mid \; r = R, \quad 0 \le \theta \le 2\pi\}, \quad R > r_{max}. \tag{2.5}$$

Let the source (charge) points be located outside of S, the fundamental solutions

$$\phi(r,\theta) = ln|\overline{PQ}|, \quad P \in S \cup \partial S \tag{2.6}$$

are harmonic, where

$$P = \{(x,y) \mid x = r\cos\theta, \; y = r\sin\theta\}. \tag{2.7}$$

Based on Bogomolny [4], the source points Q_i may be simply located uniformly on ℓ_R

$$Q_i = \{(x,y) \mid x = R\cos ih, \; y = R\sin ih\},$$

where $R > r_{max}$ and $h = \frac{2\pi}{N}$. We obtain the fundamental solutions

$$\phi_i(P) = ln|\overline{PQ_i}|, \quad i = 1, 2, \cdots, N, \tag{2.8}$$

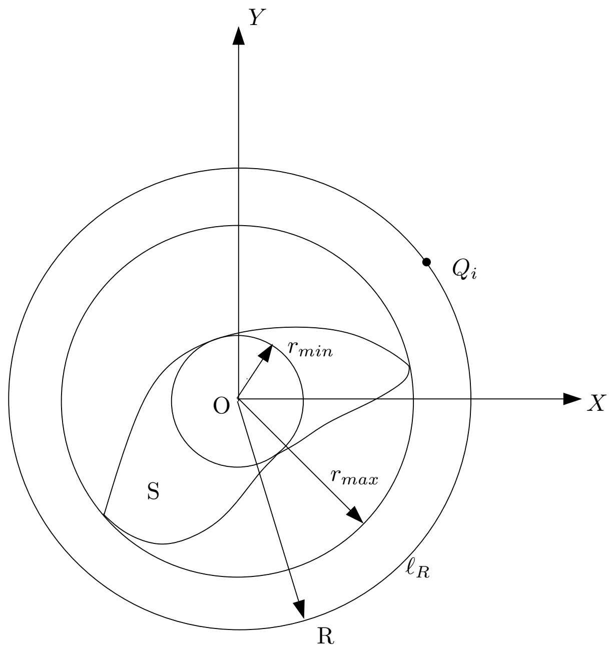

Figure 1: The solution domain S with its circles.

and the numerical solution is given by a linear combination

$$u_N = \sum_{i=1}^{N} c_i \phi_i(P) \,, \tag{2.9}$$

where c_i are the unknown coefficients to be sought. Since u_N satisfies Laplace's equation in S already, the coefficients c_i can be sought by enforcing the boundary conditions (2.2) and (2.3) only. We will follow the Trefftz method (TM) in [34, 37, 40], to seek u_N (i.e., c_i). Denote the energy

$$I(u) = \int_{\Gamma_D} (u - f)^2 + w^2 \int_{\Gamma_R} (\frac{\partial u}{\partial \nu} + \alpha u)^2, \tag{2.10}$$

where w is a positive weight. We choose $w = \frac{1}{N}$ in our computations (see [40]). Also denote by V_N the space of (2.9). Then the numerical solution u_N can be obtained by

$$I(u_N) = \min_{v \in V_N} I(v). \tag{2.11}$$

When the integrals in (2.10) involve approximation, we denote

$$\widehat{I}(v) = \widehat{\int}_{\Gamma_D} (v - f)^2 + w^2 \widehat{\int}_{\Gamma_R} (\frac{\partial v}{\partial \nu} + \alpha v)^2, \tag{2.12}$$

where $\widehat{\int}_{\Gamma_D}$ and $\widehat{\int}_{\Gamma_R}$ are the numerical approximations of $\int_{\Gamma_D}$ and $\int_{\Gamma_R}$ by some quadrature rules, such as the central or the Gaussian rule. Hence, the numerical solution $\tilde{u}_N \in V_N$ is obtained by

$$\widehat{I}(\tilde{u}_N) = \min_{v \in V_N} \widehat{I}(v). \tag{2.13}$$

We may establish the collocation equations directly from (2.2) and (2.3) to yield

$$\sum_{i=1}^{N} c_i \phi_i(P_j) = f(P_j), \quad P_j \in \Gamma_D, \tag{2.14}$$

$$\sum_{i=1}^{N} c_i \Big[\frac{\partial}{\partial \nu}\phi_i(P_j) + \alpha \phi_i(P_j)\Big] = g(P_j), \quad P_j \in \Gamma_R. \tag{2.15}$$

First, let Γ_D and Γ_R be divided into small Γ_D^j and Γ_R^j with the mesh spacings Δh_j, i.e.,

$$\Gamma_D = \bigcup_{j=1}^{M_1} \Gamma_D^j, \quad \Gamma_R = \bigcup_{j=1}^{M_2} \Gamma_R^j. \tag{2.16}$$

We obtain from (2.14) and (2.15)

$$\sqrt{\Delta h_j} \sum_{i=1}^{N} c_i \phi_i(P_j) = \sqrt{\Delta h_j}\; f(P_j), \quad P_j \in \Gamma_D, \quad j = 1, 2, \cdots, M_1 \tag{2.17}$$

$$w\sqrt{\Delta h_j} \sum_{i=1}^{N} c_i \Big\{\frac{\partial}{\partial \nu}\phi_i(P_j) + \alpha \phi_i(P_j)\Big\} = w\sqrt{\Delta h_j}\; g(P_j),$$
$$P_j \in \Gamma_R, \quad j = M_1 + 1, \cdots, M_1 + M_2, \tag{2.18}$$

where for simplicity, P_j are the midpoints of Γ_D^j and Γ_R^j. Following Lu et al. [40], Eqs (2.17) and (2.18) are just equivalent to (2.13), where the central rule is chosen for $\widehat{\int}_{\Gamma_D}$ and $\widehat{\int}_{\Gamma_R}$. In computation, we may choose the number of collocation points to be equal or larger than that of source points, i.e.,

$$M = M_1 + M_2 \geq N. \tag{2.19}$$

When the Gaussian rule is chosen, the following collocation equations are obtained.

$$\beta_i \sum_{i=1}^{N} c_i \phi_i(P_j) = \beta_j\; f(P_j), \quad P_j \in \Gamma_D, \tag{2.20}$$

$$\beta_i w \sum_{i=1}^{N} c_i \Big\{\frac{\partial}{\partial \nu}\phi_i(P_j) + \alpha \phi_i(P_j)\Big\} = \beta_j w\; g(P_j), \quad P_j \in \Gamma_R, \tag{2.21}$$

where P_j are the Gaussian nodes, the weights $\beta_j = O(\sqrt{h})$, and $h = \max_j \Delta h_j$.

3 Effective Condition Number for Least Squares Methods with Rank Deficiency

In Li et al. [36], we consider the effective condition number to follow [5, 15]. In this thesis we develop the effective condition number for least squares method with rank deficiency. Consider

$$\mathbf{F}\mathbf{x} = \mathbf{b}, \tag{3.1}$$

where $\mathbf{F} \in R^{m\times n}$ $(m \geq n)$, $\mathbf{x} \in R^n$ and $\mathbf{b} \in R^m$. When there exists the perturbation of $\mathbf{F}$ and $\mathbf{b}$, there are the equalities,

$$\mathbf{F}(\mathbf{x} + \boldsymbol{\Delta}\mathbf{x}) = \mathbf{b} + \boldsymbol{\Delta}\mathbf{b}, \tag{3.2}$$

$$(\mathbf{F} + \boldsymbol{\Delta}\mathbf{F})(\mathbf{x} + \boldsymbol{\Delta}\mathbf{x}) = \mathbf{b} + \boldsymbol{\Delta}\mathbf{b}, \tag{3.3}$$

where $\boldsymbol{\Delta}\mathbf{F} \in R^{m\times n}$, $\boldsymbol{\Delta}\mathbf{x} \in R^n$ and $\boldsymbol{\Delta}\mathbf{b} \in R^m$.

In this thesis, we assume

$$rank(\mathbf{F}) = rank(\mathbf{F} + \boldsymbol{\Delta}\mathbf{F}) = t \leq n. \tag{3.4}$$

The singular value decomposition(SVD) of $\mathbf{F}$ is given by

$$\mathbf{F} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T, \tag{3.5}$$

where $\mathbf{U} \in R^{m\times m}$ and $\mathbf{V} \in R^{n\times n}$ are orthogonal, and $\boldsymbol{\Sigma} \in R^{m\times n}$ is diagonal with t positive singular values

$$\sigma_1 \geq ... \geq \sigma_t > 0, \quad t \leq n. \tag{3.6}$$

Denote $\mathbf{U} = (u_1, ..., u_m)$ and $\mathbf{V} = (v_1, ..., v_n)$, where $u_i \in R^m$ and $v_i \in R^n$ are vectors. The solution of (3.1) is obtained by

$$\mathbf{x} = \sum_{i=1}^{t} \frac{\mathbf{u_i}^T\mathbf{b}}{\sigma_i}\mathbf{v_i}. \tag{3.7}$$

The modified condition number for (3.7) is defined in Loan [17][1]

$$\widehat{\text{Cond}} = \frac{\sigma_1}{\sigma_t}. \tag{3.8}$$

Following [36], we may define the modified effective condition number

$$\widehat{\text{Cond}}\_\text{eff} = \frac{\|\mathbf{b}\|}{\sigma_t\sqrt{(\frac{\beta_1}{\sigma_1})^2 + \cdots + (\frac{\beta_t}{\sigma_t})^2}} = \frac{\|\mathbf{b}\|}{\sigma_t\|\mathbf{x}\|}, \tag{3.9}$$

where $\beta_i = \mathbf{u_i}^T b$. When $\beta_t \neq 0$, we may also define the modified simplest effective condition,

$$\widehat{\text{Cond}}\_\text{EE} = \frac{\|\mathbf{b}\|}{\beta_t}. \tag{3.10}$$

Denote the pseudo-inverse matrix $\mathbf{F}^+$ of $\mathbf{F}$ by

$$\mathbf{F}^+ = \mathbf{V}\begin{pmatrix} \frac{1}{\sigma_1} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{\sigma_t} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}\mathbf{U^T}. \tag{3.11}$$

We cite a lemma from Lu et al. [41].

Lemma 3.1 *Let* $\mathbf{F}, \mathbf{\Delta F} \in R^{m\times n}$ *with*

$$rank(\mathbf{F}) = rank(\mathbf{F} + \mathbf{\Delta F}) = t \leq n. \tag{3.12}$$

Suppose that $\delta = \|\mathbf{F}^+\|\|\mathbf{\Delta F}\| \ll 1$, *where* $\|\cdot\|$ *is the Eclidean norm. Then there exists the bound,*

$$\frac{\|\mathbf{\Delta x}\|}{\|\mathbf{x}\|} \leq \hat{\text{Cond}}\_\text{eff}\ \frac{1}{1-\delta}\left[\frac{1+\sqrt{5}}{2}\delta + \frac{\|\mathbf{\Delta b}\|}{\|\mathbf{b}\|}\right]. \tag{3.13}$$

[1] To distinguish the effective condition numbers with rank deficiency from those with full rank, we use the notations $\widehat{\text{Cond}}$., $\widehat{\text{Cond}}\_\text{eff}$ and $\hat{\text{Cond}}\_\text{EE}$ with a hat on their head.

4 Dirichlet Problems in Disk Domains

4.1 Eigenvalues of MFS

In this and the next sections, we consider the simple disk domain S defined by

$$S = \{(r, \theta) \mid r \leq \rho,\ 0 \leq \theta \leq 2\pi\}, \tag{4.1}$$

and its circular boundary

$$\ell_\rho = \{(r, \theta) \mid r = \rho,\ 0 \leq \theta \leq 2\pi\}. \tag{4.2}$$

The source and the collocation points are uniformly located on ℓ_R and ℓ_ρ (Figure 2)

$$\begin{aligned} Q_i &= \{R \cos ih,\ R \sin ih\}, \quad i = 1, 2, \cdots, N, \\ P_i &= \{\rho \cos ih,\ \rho \sin ih\}, \quad i = 1, 2, \cdots, N, \end{aligned} \tag{4.3}$$

where $\rho < R$ and $h = \frac{2\pi}{N}$. Consider

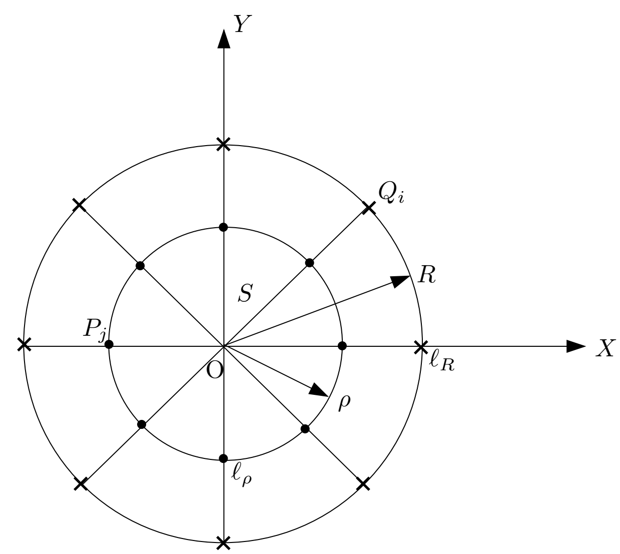

Figure 2: The locations of the source points Q_i and the collocation points P_j.

$$\Delta u = 0 \ \text{ in } \ S, \quad u = f \ \text{ on } \ \Gamma = \ell_R. \tag{4.4}$$

Choose the linear combination (2.9), the collocation equations at P_j are as in (2.14):

$$\sum_{i=1}^{N} c_i \phi_i(P_j) = f(P_j), \ \ P_j \in \ell_\rho, \tag{4.5}$$

where $\phi_i(P_j) \ = \ \ln |\overline{P_j Q_i}|$. Since

$$\begin{aligned} |\overline{P_j Q_i}|^2 &= (R\cos ih - \rho\cos jh)^2 + (R\sin ih - \rho \sin jh)^2 \\ &= R^2 + \rho^2 - 2R\rho\cos((i-j)h), \end{aligned} \tag{4.6}$$

we obtain

$$\phi_i(P_j) = \ln |\overline{P_j Q_i}| = \frac{1}{2} \ln |R^2 + \rho^2 - 2R\rho \cos((i-j)h)|.$$

We may rewrite (4.5) as the matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{4.7}$$

where $\mathbf{x} = (c_1, \cdots, c_N)^T$, $\mathbf{b} = (f(P_1), \cdots, f(P_N))^T$, and $\mathbf{A} \in R^{N \times N}$. The matrix $\mathbf{A}$ is the real circulant matrix, denoted by

$$\mathbf{A} = \begin{pmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & \cdots & a_{N-2} \\ \vdots & \vdots & \cdots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}, \tag{4.8}$$

where $a_0 = \frac{1}{2}\ln(R-\rho)^2$, and

$$a_i = \frac{1}{2} \ln |R^2 + \rho^2 - 2R\rho \cos ih|, \quad i = 1, 2, \cdots, N-1. \tag{4.9}$$

We have a lemma from Davis [17].

Lemma 4.1 *Let matrix* $\mathbf{A} = \text{Circulant}(a_0, a_1, \cdots, a_{N-1})$ *with real entries* a_i. *Then the eigenvalues of* $\mathbf{A}$ *are real, given by*

$$\lambda_k = \sum_{j=0}^{N-1} a_j \cos(kjh), \quad k = 0, 1, \cdots, N-1,$$

where $h = \frac{2\pi}{N}$*. Also, the vector* $y_0 = \frac{1}{\sqrt{N}}(1, 1, \cdots, 1)^T$ *is the eigenvector corresponding to the leading eigenvalue* λ_0*.*

Consider the integral $\int_a^b g(x)\,dx$ by the trapezoidal rule,

$$T_n(g) = h\left[\frac{1}{2}g(a) + g(a+h) + \cdots + g(a+(n-1)h) + \frac{1}{2}g(b)\right], \tag{4.10}$$

where $h = \frac{b-a}{n}$. There exists the error bound,

$$\left|\int_a^b g(x)\,dx - T_n(g)\right| = \frac{b-a}{12}h^2|g''(\eta)|, \tag{4.11}$$

where $\eta \in [a, b]$. The errors for the periodical boundary conditions are given by the following Euler-Maclaurin formulas [18, 38, 47].

Lemma 4.2 *Let* $g(x) \in C^{2k+1}[a, b]$ *satisfy the periodical boundary conditions,*

$$g'(b) = g'(a), \cdots, g^{(2k-1)}(b) = g^{(2k-1)}(a), \quad k \leq n. \tag{4.12}$$

There exist the error bounds (i.e., the Euler-Maclaurin formula)

$$\left|\int_a^b g(x)\,dx - T_n(g)\right| \leq \bar{C} M_{2k+1} h^{2k+1}, \tag{4.13}$$

where

$$M_k = \max_{x\in[a,b]} \left|g^k(x)\right|,$$

the constant

$$\bar{C} = (b-a)^{2k+2} 2^{-2k} \pi^{-2k-1} \xi(2k+1),$$

and the Riemann Zeta function

$$\xi(k) = \sum_{j=1}^{\infty} \frac{1}{j^k}.$$

We have the following theorem.

Theorem 4.1 *Let $\rho < R$. Then the eigenvalues of matrix $\mathbf{A}$ in (4.8) are given by*

$$\lambda_0 = N \ln R + \varepsilon, \tag{4.14}$$

$$\lambda_k = -\frac{N}{2k}\left(\frac{\rho}{R}\right)^k + \varepsilon, \quad k = 1, 2, \cdots, N-1, \tag{4.15}$$

where $\varepsilon = O(h^{2N})$, and $h = \frac{1}{N}$.

Proof : From Lemma 4.1, we have the eigenvalues of $\mathbf{A}$ as

$$\lambda_k = \sum_{j=0}^{N-1} a_j \cos(kjh). \tag{4.16}$$

Denote

$$a = \frac{R}{\rho} > 1. \tag{4.17}$$

We obtain from (4.16) and (4.9)

$$\begin{aligned}\lambda_0 &= \sum_{j=0}^{N-1} a_j = \frac{1}{2}\sum_{j=0}^{N-1} \ln|R^2 + \rho^2 - 2R\rho\cos jh| \\ &= N\ln\rho + \frac{1}{2h}\sum_{j=0}^{N-1} h\ln(1 + a^2 - 2a\cos jh).\end{aligned} \tag{4.18}$$

Let $g(x) = \ln(1 + a^2 - 2a\cos x)$. Then the function $g(x) \in C^\infty[0, 2\pi]$ for (4.17), and satisfies $g^{(k)}(0) = g^{(k)}(2\pi), \quad k = 0, 1, \cdots$. From Lemma 4.2, we have

$$\sum_{j=1}^{N-1} h\ln(1 + a^2 - 2a\cos jh) = \int_0^{2\pi} \ln(1 + a^2 - 2a\cos x)dx + O(h^{2N+1}). \tag{4.19}$$

From Gradshteyn and Ryzhik [22], p 527, there exists the integral formula,

$$\int_0^{2\pi} \ln(1 + a^2 - 2a\cos x)dx = \begin{cases} 0 & (a^2 < 1), \\ 2\pi\ln a^2 & (a^2 > 1). \end{cases} \tag{4.20}$$

Combining (4.19) and (4.20) yields

$$\sum_{j=0}^{N-1} h\ln(1+a^2-2a\cos jh) = 4\pi\ln a + O(h^{2N+1}). \tag{4.21}$$

Hence we obtain from $h = \frac{2\pi}{N}$, (4.17), (4.18) and (4.21)

$$\lambda_0 = N\ln\rho + \frac{1}{2h}4\pi\ln a + O(h^{2N+1}) = N\ln R + O(h^{2N}).$$

This is the first result (4.14).

Next for λ_k in (4.15), we have

$$\begin{aligned}\lambda_k &= \frac{1}{2}\sum_{j=0}^{N-1}\ln(R^2+\rho^2-2R\rho\cos jh)\cos kjh \\ &= \frac{\ln\rho}{h}\sum_{j=0}^{N-1} h\cos kjh + \frac{1}{2h}\sum_{j=1}^{N} h\ln(1+a^2-2a\cos jh)\cos kjh\,. \end{aligned} \tag{4.22}$$

There exist the equalities,

$$\sum_{j=0}^{N-1} h\cos kjh = h\,Re\left(\sum_{j=0}^{N-1} e^{ikj\frac{2\pi}{N}}\right) = 0, \tag{4.23}$$

where we have used

$$1+a+\cdots+a^{N-1} = \frac{1-a^N}{1-a}.$$

From Lemma 4.2,

$$\begin{aligned}&\sum_{j=1}^{N} h\ln(1+a^2-2a\cos jh)\cos kjh \\ &= \int_0^{2\pi}\ln(1+a^2-2a\cos x)\cos kx\,dx + O(h^{2N+1}).\end{aligned} \tag{4.24}$$

There exists the integral formula in Gradshteyn and Ryzhik [22], p 593,

$$\int_0^{2\pi}\ln(1+a^2-2a\cos x)\cos kx\,dx = \begin{cases} -\frac{2\pi}{k}a^k, & a^2<1, \\ -\frac{2\pi}{ka^k}, & a^2>1. \end{cases} \tag{4.25}$$

Combining (4.24) and (4.25) yields

$$\sum_{j=1}^{N} h \ln(1 + a^2 - 2a \cos jh) \cos kjh = -\frac{2\pi}{ka^k} + O(h^{2N+1}). \tag{4.26}$$

Hence, we obtain from (4.22), (4.23) and (4.26)

$$\lambda_k = -\frac{1}{2h}\left[\frac{2\pi}{ka^k} + O(h^{2N+1})\right] = -\frac{N}{2k}\left(\frac{\rho}{R}\right)^k + O(h^{2N}), \quad k = 1, 2, \cdots, N-1. \tag{4.27}$$

This is the second result (4.15), and completes the proof of Theorem 4.1 ∎

The traditional condition number of $\mathbf{A}$ with full rank is defined by

$$\text{Cond} = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|}, \tag{4.28}$$

where λ_i are the eigenvalues of $\mathbf{A}$. Since $\varepsilon = O(N^{2N})$ is much smaller than $\frac{N}{k}\left(\frac{\rho}{R}\right)^k$, we have the following corollary.

Corollary 4.1 *Let $\rho < R$ and $R \neq 1$, there exists the bound*

$$\text{Cond} \le C_1 \; N \left(\frac{R}{\rho}\right)^{N-1}, \tag{4.29}$$

where the constant

$$C_1 = \max\left\{|\ln R|, \; \frac{\rho}{R}\right\}. \tag{4.30}$$

Proof : Form Theorem 4.1, we have

$$\max_i |\lambda_i| = \max\{|\lambda_0|, \, |\lambda_1|\} \le C \max\{N|\ln R|, \; N\left(\frac{\rho}{R}\right)\}, \tag{4.31}$$

and

$$\min_i |\lambda_i| = |\lambda_{N-1}| = \frac{N}{2(N-1)}\left(\frac{\rho}{R}\right)^{N-1} + \varepsilon \ge \left(\frac{\rho}{R}\right)^{N-1}, \tag{4.32}$$

where we have used the following bound,

$$\frac{N}{2(N-1)} \geq 1 \quad \text{for } N \geq 2.$$

Combining (4.28), (4.31) and (4.32) yields the desired result (4.30), and the proof of Corollary 4.1 is completed. ∎

Now, we also give the modified condition number for $R = 1$.

Corollary 4.2 *Let $\rho < R = 1$, there exists the bound,*

$$\widehat{\text{Cond}} = \frac{|\lambda_1|}{|\lambda_{N-1}|} \approx N\rho^{-N+2}\,.$$

4.2 General solutions by MFS for the Case of $\rho \leq R = 1$

When $\rho \leq R = 1$, the solution in (2.9) is reduced to

$$u_N = \sum_{i=1}^{N} c_i \psi_i(\theta)\,, \tag{4.33}$$

where

$$\psi_i(\theta) = \frac{1}{2}\ln(1 + \rho^2 - 2\rho\cos(\theta - ih)),$$

and $h = \frac{2\pi}{N}$. Since the circulant $\mathbf{A}$ is very singular, see (4.14) as $R = 1$, we obtain the solution from (3.7)

$$\mathbf{x}_{\text{LSM}} = \sum_{i=0}^{N-1} \frac{\mathbf{u_i}^T\mathbf{b}}{\sigma_i}\mathbf{v_i}, \tag{4.34}$$

where $\sigma_i = |\lambda_i|$ and $\mathbf{x}_{\text{LSM}} = (\bar{c}_1, \cdots, \bar{c}_N)^T$. From Lemma 4.1, $(1, \cdots, 1)^T$ is the eigenvector corresponding to $\lambda_0 = 0$. Then the general solution of $\mathbf{x}$ is given by

$$u_N = \sum_{i=1}^{N} c_i \psi_i(\theta) + c \sum_{i=1}^{N} \psi_i(\theta), \tag{4.35}$$

where c is an arbitrary constant. Since there exists the equality

$$\sum_{i=1}^{N} \psi_i(\theta) = \varepsilon, \tag{4.36}$$

shown below, Eq.(4.35) gives the unique solution

$$u_N = \sum_{i=1}^{N} \bar{c}_i \psi_i(\theta) + \varepsilon. \tag{4.37}$$

Now we prove (4.36). For the given (ρ, θ), we have from Lemma 4.2,

$$\begin{aligned}
\sum_{i=1}^{N} \psi_i(\theta) &= \frac{1}{2h} \sum_{i=1}^{N} h \ln(1 + \rho^2 - 2\rho\cos(\theta - ih)) \\
&= \frac{1}{2h} \int_0^{2\pi} \ln(1 + \rho^2 - 2\rho\cos(\phi - \theta)) d\phi + \varepsilon \\
&= \frac{1}{2h} \int_{-\theta}^{2\pi-\theta} \ln(1 + \rho^2 - 2\rho\cos t) dt + \varepsilon \\
&= \frac{1}{2h} \int_0^{2\pi} \ln(1 + \rho^2 - 2\rho\cos t) dt = \varepsilon,
\end{aligned}$$

where we have used (4.20) with $\rho^2 < 1$.

Note that since (4.37) is valid for any (ρ, θ) with $\rho < R = 1$, the unique solution (4.37) can be denoted by

$$u_N = \sum_{i=1}^{N} \bar{c}_i \psi_i(\theta)\,, \tag{4.38}$$

with an infinite small ε for the bounded domain S with $r_{max} < R = 1$. Based on the analysis for $R = 1$ in this subsection, the difficulty of $R = 1$ is fake, and it can be bypassed by using the SVD in Section 3. In fact, the non-unique solutions can also be removed for BIE, see Chirstiansen [14].

4.3 Linkage to Methods of Boundary Integral Equations of First Kind

Comparisons between MFS and the method of boundary integral equation (BIE) of first kind are made in [13]. In this subsection, we focus on magnitude of coefficients c_i.

The Dirichlet problem of Laplace's equation can be converted to the following boundary integral equation

$$u(P) = -\frac{1}{2\pi}\int_\Gamma \sigma(Q)\ln|\overline{PQ}|\,d\ell_Q, \quad P \in S \cup \partial S, \tag{4.39}$$

where $\sigma(Q)$ is the single layer density. Suppose that the logarithmic capacity $C_\Gamma \neq 1$ (i.e., $\rho \neq 1$ for $\Gamma = \ell_\rho$), there exists a unique solution of (4.39). When $P \in \Gamma$, we have

$$-\frac{1}{2\pi}\int_\Gamma \sigma(Q)\ln|\overline{PQ}|\,d\ell_Q = f(P). \tag{4.40}$$

Consider $\Gamma = \ell_R$, and choose

$$P_i = Q_i = (\rho\cos ih, \rho\sin ih), \quad h = \frac{2\pi}{N}. \tag{4.41}$$

Then the mechanic quadrature method (MQM) is given in Huang et al. [28], as

$$-\frac{1}{2\pi}\left\{h\sum_{\substack{i=1\\ i\neq j}}^{N}\sigma(Q_i)\ln|\overline{Q_iQ_j}| + \sigma(Q_j)\ln\frac{h}{2\pi}\right\} = f(Q_j), \quad j = 1, 2, \cdots, N, \tag{4.42}$$

where $\ln|\overline{Q_iQ_j}| = \ln\left|2\rho\sin\frac{(i-j)h}{2}\right|$. In [28] the errors of $\sigma(Q_i)$ obtained from (4.42) are $O(h^3)$, and the condition number of the stiffness matrix from (4.42) is given by

$$\text{Cond} = O(N). \tag{4.43}$$

Note that Eq. (4.43) can also be obtained from (4.29) as $R \longrightarrow \rho$.

Let us compare (4.42) with the MFS in (4.5). When $R \longrightarrow \rho$ and $P_i \longrightarrow Q_i$, it seems to have the approximate relation

$$\bar{c}_i \approx -\frac{h}{2\pi}\sigma(Q_i). \tag{4.44}$$

Hence we have

$$\|\mathbf{x}_{\text{LSM}}\|^2 = \sum_{i=1}^{N}\bar{c}_i^2 \approx \frac{h}{4\pi^2\rho}\sum_{i=1}^{N}h\rho\sigma^2(Q_i) \approx \frac{h}{4\pi^2\rho}\int_\Gamma \sigma^2(Q)d\ell_Q. \tag{4.45}$$

Since the density function $\sigma(x) = \frac{\partial u^-}{\partial \nu} - \frac{\partial u^+}{\partial \nu}$ is bounded for the smooth solution, we have from (4.45)

$$\|\mathbf{x}_{\mathrm{LSM}}\| \approx \frac{\sqrt{h}}{2\pi\sqrt{\rho}}\|\sigma\|_{0,\Gamma} = O\left(\frac{1}{\sqrt{N}}\right). \tag{4.46}$$

The equation (4.46) implies that the ideal solution $\mathbf{x}_{\mathrm{LSM}}$ for $P_i \longrightarrow Q_i$ is expected to be as small as $o(1)$.

5 Neumann and Robin Problems on Disk Domains

5.1 Eigenvalues of MFS for Neumann Problems

Consider the Neumann problem

$$-\Delta u = 0 \;\text{ in } S, \qquad \frac{\partial u}{\partial \nu} = g \;\text{ on } \partial S, \tag{5.1}$$

where $\int_{\partial S} g \, d\ell = 0$ is the consistent condition to guarantee the existence of the solutions of (5.1). Then the general solutions of (5.1) are given by

$$u_N^G = c + \sum_{i=1}^{N} c_i \phi_i(P), \tag{5.2}$$

where $\phi_i = \ln|\overline{PQ_i}|$, Q_i are located uniformly on ℓ_R as in (2.8), and c_i are the expansion coefficients. In (5.2), c is an arbitrary constant. In our computation, we may find a solution

$$u_N = \sum_{i=1}^{N} c_i \phi_i(P), \tag{5.3}$$

then the general solutions are obtained from (5.3) by adding an arbitrary constant c. By the CTM in Section 2, $\tilde{u}_N$ can be obtained by

$$\widehat{T}(\tilde{u}_N) = \min_{v \in V_N} \widehat{T}(v), \tag{5.4}$$

where V_N is the set of the functions (5.3), and

$$\widehat{T}(v) = \widehat{\int_{\partial S}} \left(\frac{\partial u}{\partial \nu} - g\right)^2 .$$

On the other hand, we may also formulate the collocation equations directly

$$\sum_{i=1}^{N} c_i \frac{\partial}{\partial \nu} \phi_i(P_j) = g(P_j), \quad j = 1, 2, \cdots, N, \tag{5.5}$$

where $\phi_i(P_j) = \ln |\overline{Q_i P_j}|$, Q_i and P_j are given in (4.3) and $R > \rho$. Hence we have

$$\frac{\partial}{\partial \nu} \phi_i(P_j) = \frac{\partial}{\partial r} \ln |\overline{Q_i P_j}| = \frac{1}{|\overline{Q_i P_j}|} \cos(\vec{r}, \overrightarrow{Q_i P_j}), \tag{5.6}$$

where

$$\vec{r} = \overrightarrow{OP_j} = \rho \cos jh \ \vec{i} + \rho \sin jh \ \vec{j} \ , \tag{5.7}$$

$$\overrightarrow{Q_i P_j} = (\rho \cos jh - R \cos ih)\vec{i} + (\rho \sin jh - R \sin ih)\vec{j} \ , \tag{5.8}$$

$h = \frac{2\pi}{N}$, and $\vec{i}$ and $\vec{j}$ are the unit vectors along x and y directions, respectively. Moreover there exists the equality

$$\begin{aligned} \cos(\vec{r}, \overrightarrow{Q_i P_j}) &= \frac{(\vec{r}, \overrightarrow{Q_i P_j})}{|\vec{r}||\overrightarrow{Q_i P_j}|} \\ &= \frac{1}{\rho |\overline{Q_i P_j}|} \{\rho \cos jh(\rho \cos jh - R \cos ih) + \rho \sin jh(\rho \sin jh - R \sin ih)\} \\ &= \frac{1}{\rho |\overline{Q_i P_j}|} \left\{\rho^2 - \rho R \cos(j-i)h\right\}. \end{aligned} \tag{5.9}$$

Then the derivatives (5.6) lead to

$$\begin{aligned} \frac{\partial}{\partial \nu} \phi_i(P_j) &= -\frac{\rho R \cos(j-i)h - \rho^2}{\rho \{R^2 + \rho^2 - 2R\rho \cos(j-i)h\}} \\ &= -\frac{a \cos(j-i)h - 1}{\rho(a^2 + 1 - 2a \cos(j-i)h)} \ , \end{aligned} \tag{5.10}$$

where $a = \frac{R}{\rho} > 1$. Hence Eqs. (5.5) can also be denoted by

$$\mathbf{D}\mathbf{x} = \mathbf{b}, \tag{5.11}$$

where $\mathbf{x} = (c_1, \cdots, c_n)^T$, $\mathbf{b} \in (g(P_1), \cdots, g(P_N))^T$, and $\mathbf{D} \in R^{N \times N}$. The matrix $\mathbf{D}$ is also the circulant matrix

$$\mathbf{D} = \text{circulant}(d_0, d_1, \cdots, d_{N-1}), \tag{5.12}$$

where

$$d_0 = \frac{1}{\rho(1-a)}, \tag{5.13}$$

$$d_k = -\frac{a\cos kh - 1}{\rho(1+a^2-2a\cos kh)}, \quad k = 1, 2, \cdots, N-1. \tag{5.14}$$

We have the following theorem.

Theorem 5.1 *For the Neumann problem (5.1) on the disk S, the eigenvalues of matrix* $\mathbf{D}$ *from the* MFS *in (5.5) are given by*

$$\lambda_0 = \varepsilon \tag{5.15}$$

$$\lambda_k = -\frac{N}{2(R+\rho)}\frac{1}{a^k} + \varepsilon, \quad k = 1, 2, \cdots N-1, \tag{5.16}$$

where $a = \frac{R}{\rho} > 1$, and $\varepsilon = O(h^{2N})$.

Proof : Form Lemma 4.1, the eigenvalues of the circulant matrix $\mathbf{D}$ are given by

$$\lambda_k = \sum_{j=0}^{N-1} d_j \cos kjh, \quad k = 0, 1, \cdots, N-1. \tag{5.17}$$

First we have from Lemma 4.2

$$\begin{aligned}\lambda_0 &= \sum_{j=0}^{N-1} d_j = -\sum_{j=0}^{N-1} \frac{a\cos jh - 1}{\rho(1+a^2-2a\cos jh)} \\ &= -\frac{1}{\rho h}\left[\int_0^{2\pi} \frac{a\cos x - 1}{1+a^2-2a\cos x}dx + O(h^{2N+1})\right].\end{aligned} \tag{5.18}$$

Since

$$\frac{a\cos x - 1}{1+a^2-2a\cos x} = -\frac{1}{2} + \frac{a^2-1}{2}\frac{1}{1+a^2-2a\cos x},$$

Eq. (5.18) leads to

$$\lambda_0 = \frac{1}{\rho h}\left\{\pi - \frac{a^2-1}{2}\int_0^{2\pi} \frac{1}{1+a^2-2a\cos x}\,dx\right\} + O(h^{2N}). \tag{5.19}$$

There exists the integration formula in [22], p. 367,

$$\int_0^{2\pi} \frac{\cos kx}{1+a^2-2a\cos x}\,dx = \begin{cases} \dfrac{2\pi a^k}{1-a^2}\ , & a^2<1, \\[2ex] \dfrac{2\pi}{(a^2-1)}\dfrac{1}{a^k}\ , & a^2>1. \end{cases} \tag{5.20}$$

Hence we have (5.19) and (5.20),

$$\lambda_0 = -\frac{1}{\rho h}\left\{\pi - \frac{a^2-1}{2}\cdot\frac{2\pi}{a^2-1}\right\} + O(h^{2N}) = \varepsilon. \tag{5.21}$$

This is the first desired result (5.15).

Next, we have from (5.17) and Lemma 4.2

$$\lambda_k = -\sum_{j=0}^{N-1}\frac{\cos kjh(a\cos jh-1)}{\rho(1+a^2-2a\cos jh)} = -\frac{1}{\rho h}\int_0^{2\pi}\frac{\cos kx(a\cos x-1)}{1+a^2-2a\cos x}\,dx + \varepsilon. \tag{5.22}$$

Since

$$\cos kx\ \cos x = \frac{1}{2}(\cos(k+1)x + \cos(k-1)x),$$

we have from (5.22) and (5.20),

$$\begin{aligned} \lambda_k &= \frac{1}{\rho h}\left\{-\frac{a}{2}\int_0^{2\pi}\frac{\cos(k+1)x+\cos(k-1)x}{1+a^2-2a\cos x}\,dx + \int_0^{2\pi}\frac{\cos kx}{1+a^2-2\cos x}\,dx\right\} + \varepsilon \\ &= \frac{1}{\rho h}\left\{-\frac{a}{2}\,\frac{2\pi}{a^2-1}\left(\frac{1}{a^{k+1}}+\frac{1}{a^{k-1}}\right) + \frac{2\pi}{a^2-1}\,\frac{1}{a^k}\right\} + \varepsilon \\ &= -\frac{1}{\rho h}\frac{\pi}{a^k} + \varepsilon = -\frac{N}{2\rho a^k} + \varepsilon. \end{aligned} \tag{5.23}$$

This is the second desired result (5.16), and completes the proof of Theorem 5.1 ∎

Corollary 5.1 *Let the conditions in Theorem 5.1 hold. By the TSVD in Section 3, the condition number is given by*

$$\widehat{\text{C}}\text{ond} = \frac{|\lambda_1|}{|\lambda_N|} \le C\left(\frac{R}{\rho}\right)^{N-2}. \tag{5.24}$$

Below, let us discuss the general solutions (5.2). From Lemma 4.1, the general solution vector is given by

$$\mathbf{x}^G = \mathbf{x}_{\mathrm{LSM}} + c\,(1, 1, \cdots, 1)^T,$$

where c is an arbitrary constant, $(1, 1, \cdots, 1)^T$ is the eigenvector corresponding to $\lambda_0 = 0$, and

$$\mathbf{x}_{\mathrm{LSM}} = (\bar{c}_1, \cdots, \bar{c}_N)^T = \sum_{i=1}^{N-1} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i, \tag{5.25}$$

with $\sigma_i = |\lambda_i|$. Hence, the general solutions (5.2) are reduced to

$$u_N = c + \beta \sum_{i=1}^{N} \phi_i(P) + \sum_{i=1}^{N} \bar{c}_i \phi_i(P), \tag{5.26}$$

where c and β are two arbitrary constants. Since for $P \in (S \cup \partial S)$,

$$\sum_{i=1}^{N} \phi_i(P) = N \ln R + \varepsilon, \tag{5.27}$$

we have from (5.26)

$$u_N = c + \sum_{i=1}^{N} \bar{c}_i \phi_i(P), \tag{5.28}$$

where c is a constant independent of P.

Next we show (5.27). In fact,

$$\begin{aligned}\sum_{i=1}^{N} \phi_i(P) = \sum_{i=1}^{N} \ln |PQ_i| = \frac{1}{2} \sum_{i=1}^{N} \ln(R^2 + \rho^2 - 2R\rho \cos(\theta - ih))\\ = N \ln \rho + \frac{1}{2h} \sum_{i=1}^{N} h(1 + a^2 - 2a \cos(\theta - ih)),\end{aligned} \tag{5.29}$$

where $a = \frac{R}{\rho} > 1$. We have from Lemma 4.2 and (4.20),

$$\sum_{i=1}^{N} \phi_i(P) \;\; = \;\; N \ln \rho + \frac{1}{2h} \int_0^{2\pi} \ln(1 + a^2 - 2a \cos(\theta - \phi))\, d\phi + \varepsilon$$

$$
\begin{aligned}
&= N \ln \rho + \frac{1}{2h} \int_{-\theta}^{2\pi-\theta} (1 + a^2 - 2a \cos t)\, dt + \varepsilon \\
&= N \ln \rho + \frac{1}{2h} \int_{0}^{2\pi} (1 + a^2 - 2a \cos t)\, dt + \varepsilon \qquad (5.30) \\
&= N \ln \rho + \frac{1}{2h} 4\pi \ln a + \varepsilon \;=\; N \ln R + \varepsilon.
\end{aligned}
$$

This is (5.27). Hence for any Neumann problem in boundary domain, the general solutions are also given in (5.28).

5.2 Modification of MFS for Neumann Problems

Note that the minimal singular value $\sigma_{min}(\mathbf{D})$ of the matrix $\mathbf{D}$ in (5.11) is infinitesmall, but not exactly zero. The solution (5.25) is still valid, which is, indeed, the truncated singular value decomposition [**?**].

However, the operator of the Neumann problem (5.1) is singular. Its discrete matrix should also be singular. In this subsection, we propose a modification of MFS, called MMFS such that the minimal singular value $\sigma_{min}(\hat{\mathbf{D}}) = 0$, where the matrix $\hat{\mathbf{D}}$ is a slightly modification of $\mathbf{D}$. We shall prove that the solution of MMFS is very close to the solution (5.25).

For a given N, from (5.15) and (5.18) we have

$$
\sigma_{min}(\mathbf{D}) = \lambda_0 = d_0 + \sum_{k=1}^{N-1} d_k = \bar{\varepsilon}\,, \qquad (5.31)
$$

where $\bar{\varepsilon}$ is a constant and

$$
\bar{\varepsilon} = O(h^{2N})\,. \qquad (5.32)
$$

Denote a parameter β such that

$$
\beta d_0 + \sum_{k=1}^{N-1} d_k = 0\,, \qquad (5.33)
$$

where d_0 and d_k are the coefficients given in (5.13) and (5.14) respectively. Hence we have from (5.31)

$$
\beta \;=\; -\frac{1}{d_0} \sum_{k=1}^{N-1} d_k \;=\; -\frac{1}{d_0} \left[\bar{\varepsilon} - d_0 \right] = 1 - \frac{\bar{\varepsilon}}{d_0}\,, \qquad (5.34)
$$

and

$$d_0(1-\beta) = \bar{\varepsilon}\,. \tag{5.35}$$

The parameter β is very close to 1.

The circulant $\mathbf{D}$ in (5.12) may be replaced by

$$\hat{\mathbf{D}} = \text{circulant}(\beta d_0, d_1, \cdots, d_{N-1})\,. \tag{5.36}$$

By following the arguments in Section 5.2, we have

$$\hat{\sigma}_{min} = \sigma_{min}(\hat{\mathbf{D}}) = \beta d_0 + \sum_{i=1}^{N-1} d_i = 0\,. \tag{5.37}$$

From (5.12) and (5.36), we obtain

$$\hat{\mathbf{D}} = \mathbf{D} + (\beta - 1)d_0\mathbf{I} = \mathbf{D} - \bar{\varepsilon}\,\mathbf{I}\,, \tag{5.38}$$

where $\mathbf{I} \in R^{n\times n}$ is the identity matrix. Denote the singular values of $\hat{\mathbf{D}}$ by $\hat{\sigma}_i = \sigma_i(\hat{\mathbf{D}})$. Then we have from (5.38)

$$\hat{\sigma}_i = \sigma_i - \bar{\varepsilon}\,. \tag{5.39}$$

Hence we may replace (5.11) by seeking the solution

$$\hat{\mathbf{D}}\hat{\mathbf{x}}_{LSM} = \mathbf{b}\,, \tag{5.40}$$

and obtain the solution

$$\hat{\mathbf{x}}_{LSM} = \sum_{i=1}^{N-1} \frac{\beta_i}{\hat{\sigma}_i}\mathbf{v}_i\,, \tag{5.41}$$

where $\beta_i = \mathbf{u}_i^T\mathbf{b}$.

We have the following theorem.

Theorem 5.2 *Let $a = \frac{R}{\rho} > 1$, and assume that the vector $\mathbf{b}$ in (5.40) satisfies the consistent condition*

$$\sum_{i=1}^{N} \mathbf{b}_i = 0\,. \tag{5.42}$$

Then for the solution $\hat{\mathbf{x}}_{LSM}$ by the MMFS, there exists the bound,

$$\frac{\|\hat{\mathbf{x}}_{LSM} - \mathbf{x}_{LSM}\|}{\|\mathbf{x}_{LSM}\|} \leq \frac{C\bar{\varepsilon}}{\sigma_{N-1}}\,, \tag{5.43}$$

where $\mathbf{x}_{LSM}$ is given in (5.25), $\sigma_{N-1} = |\lambda_{N-1}|$, λ_{N-1} is given in (5.16), and C is a constant independent of N.

Proof : We have

$$\|\hat{\mathbf{x}}_{LSM} - \mathbf{x}_{LSM}\|^2 = \sum_{i=1}^{N} \beta_i^2 \left[\frac{1}{\hat{\sigma}_i} - \frac{1}{\sigma_i}\right]^2 . \tag{5.44}$$

Since the eigenvector $\mathbf{u}_{min}\frac{1}{\sqrt{N}}(1, 1, \cdots, 1)^T$, we obtain from (5.42)

$$\beta_n = \mathbf{u}_{min}^T \mathbf{b} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \mathbf{b}_i = 0\,. \tag{5.45}$$

From (5.39),

$$\frac{1}{\hat{\sigma}_i} - \frac{1}{\sigma_i} = \frac{\sigma_i - \hat{\sigma}_i}{\hat{\sigma}_i \sigma_i} = \frac{\bar{\varepsilon}}{\hat{\sigma}_i \sigma_i}\,. \tag{5.46}$$

Then Eq. (5.44) is reduced to

$$\begin{aligned} \|\hat{\mathbf{x}}_{LSM} - \mathbf{x}_{LSM}\|^2 &= \sum_{i=1}^{N} \beta_i^2 \left[\frac{1}{\hat{\sigma}_i} - \frac{1}{\sigma_i}\right]^2 = (\bar{\varepsilon})^2 \sum_{i=1}^{N-1} \left(\frac{\beta_i^2}{\hat{\sigma}_i^2 \sigma_i^2}\right) \leq \frac{(\bar{\varepsilon})^2}{\hat{\sigma}_{N-1}^2} \sum_{i=1}^{N-1} \frac{\beta_i^2}{\sigma_i^2} \\ &\leq C \frac{\bar{\varepsilon}^2}{\sigma_{N-1}^2} \|\mathbf{x}_{LSM}\|^2\,, \end{aligned} \tag{5.47}$$

where we have used

$$\hat{\sigma}_{N-1} = \sigma_{N-1} - \bar{\varepsilon} \approx \sigma_{N-1}\,. \tag{5.48}$$

The desired result (5.44) follows from (5.47). This completes the proof of Theorem 5.2. ∎

5.3 Eigenvalues of MFS for Robin Problems

Consider the Robin problem

$$\Delta u = 0 \quad \text{in } S\ , \qquad \frac{\partial u}{\partial \nu} + \alpha u = g \quad \text{on } \Gamma = \ell_p\ , \tag{5.49}$$

where $\alpha(> 0)$ is a constant. Choose the admissible functions (5.3), use the MFS in (2.15), and obtain the matrix equation (5.11), where $\mathbf{D}$ is also the circulant matrix,

$$\mathbf{D} = \text{ circulant}\,(\bar{d}_0, \bar{d}_1, \cdots, \bar{d}_{N-1}), \tag{5.50}$$

where

$$\bar{d}_k = -\sum_{j=0}^{N-1} \frac{a\cos kjh - 1}{\rho(1+a^2-2a\cos jh)} + \alpha\sum_{j=0}^{N-1}\left\{\ln\rho + \frac{1}{2}\ln(1+a^2-2a\cos jh)\right\}, \tag{5.51}$$

with $a = \frac{R}{\rho} > 0$. Based on Lemma 4.1, the eigenvalues of matrix $\mathbf{D}$ in (5.50) are given by

$$\begin{aligned}\lambda_k = &-\sum_{j=0}^{N-1} \frac{\cos(kjh)(a\cos jh - 1)}{\rho(1+a^2-2a\cos jh)} \\ &+\alpha\sum_{j=0}^{N-1}\cos(kjh)\left\{\ln\rho + \frac{1}{2}\ln(1+a^2-2a\cos jh)\right\}.\end{aligned} \tag{5.52}$$

By following the proof of Theorems 4.1 and 5.1, we obtain the following theorem.

Theorem 5.3 *For the Robin problem by the MFS in (2.15), the eigenvalues of matrix* $\mathbf{D}$ *in (5.50) are given by*

$$\begin{aligned}\lambda_0 &= \alpha N\ln R + \varepsilon, \\ \lambda_k &= -\left(\frac{\rho}{R}\right)^k\left(\frac{N}{2\rho} + \frac{\alpha N}{2k}\right) + \varepsilon, \quad k = 1, 2, \cdots, N-1,\end{aligned} \tag{5.53}$$

where $\varepsilon = O(h^{2N})$.

Corollary 5.2 *Let $R \neq 1$ and the conditions in Theorem 5.3 hold. The condition number of matrix* $\mathbf{D}$ *has the bound,*

$$\text{Cond} \leq \bar{C} \left(\frac{R}{\rho}\right)^{N-1}, \tag{5.54}$$

where the constant

$$\bar{C} = \frac{\max\left\{2\alpha|\ln R|,\ \frac{\rho}{R}\left(\frac{1}{\rho} + \alpha\right)\right\}}{\frac{1}{\rho} + \frac{\alpha}{2N}}. \tag{5.55}$$

When $\alpha \longrightarrow 0$ and $\alpha \longrightarrow \infty$, we have $\bar{C} = O(1)$ and $\bar{C} = O(N)$. Corollary 5.2 leads to Corollaries 5.1 and 4.1, respectively. When $R = 1$, the discussions on the MFS may follow Sections 4.2 and 5.1.

6 Mixed Problems in Bounded and Simply-Connected Domains

6.1 Trefftz Methods

Consider the mixed problem (2.1) – (2.3), where S is a bounded and simply-connected domain. When $\Gamma_R = \emptyset$ and $\Gamma_D = \emptyset$, Eqs. (2.1) – (2.3) denote the Dirichlet and the Robin problems, respectively. Specially, when $\Gamma_D = \emptyset$ and $\alpha = 0$, Eqs. (2.1) – (2.3) also denote the Neumann problem. First, we study the stability of the CTM (2.11) without integration approximation, and obtain the normal equation

$$\mathbf{B}\mathbf{x} = \mathbf{b}, \tag{6.1}$$

where $\mathbf{x} = (c_1, \cdots, c_N)^T$, $\mathbf{b} \in R^N$ is the known vector, and $\mathbf{B} \in R^{N\times N}$. The matrix $\mathbf{B}$ is symmetric and positive definite, given by

$$\frac{1}{2}\mathbf{x}^T\mathbf{B}\mathbf{x} = I_0(v),$$

where $v \in V_N$ and

$$I_0(v) = \int_{\Gamma_D} v^2 + w^2 \int_{\Gamma_R} (\frac{\partial v}{\partial \nu} + \alpha v)^2, \tag{6.2}$$

$w \in [0,1]$ and $\alpha \geq 0$. Hence the maximal and the minimal eigenvalues of $\mathbf{B}$ can be obtained by

$$\lambda_{max}(\mathbf{B}) = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T\mathbf{B}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \max_{\substack{\mathbf{x}\neq 0\\ v\in V_N}} \frac{2\, I_0(v)}{\|\mathbf{x}\|^2}\,, \tag{6.3}$$

and

$$\lambda_{min}(\mathbf{B}) = \min_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T\mathbf{B}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \min_{\substack{\mathbf{x}\neq 0\\ v\in V_N}} \frac{2\, I_0(v)}{\|\mathbf{x}\|^2}. \tag{6.4}$$

When $R > r_{max}$, since $\phi_i(P)$ and $\frac{\partial \phi_i}{\partial \nu}(P)$ are bounded on Γ, we have from the Schwarz inequality

$$\begin{aligned} \int_{\Gamma_D} v^2 &= \int_{\Gamma_D} \left(\sum_{i=1}^N c_i\phi_i(P)\right)^2 \leq \left(\sum_{i=1}^N |c_i| \int_{\Gamma_D} |\phi_i(P)|\right)^2 \\ &\leq C\left(\sum_{i=1}^N |c_i|\right)^2 \leq CN\sum_{i=1}^N c_i^2 = CN\|\mathbf{x}\|^2, \end{aligned} \tag{6.5}$$

and from the bound α,

$$\begin{aligned} \int_{\Gamma_R} (\frac{\partial v}{\partial \nu} + \alpha v)^2 &= \int_{\Gamma_R} \left\{\sum_{i=1}^N c_i \left(\frac{\partial \phi_i}{\partial \nu}(P) + \alpha\phi_i(P)\right)\right\}^2 \\ &\leq C\left(\sum_{i=1}^N |c_i|\right)^2 \leq CN\|\mathbf{x}\|^2, \end{aligned} \tag{6.6}$$

where C is a constant independent of N. From (6.2), (6.3) and $w = \frac{1}{N} \leq 1$, we obtain

$$\lambda_{max}(\mathbf{B}) = \max_{\substack{\mathbf{x}\neq \mathbf{0}\\ v\in V_n}} \left\{\frac{\int_{\Gamma_D} v^2}{\|\mathbf{x}\|^2} + w^2\frac{\int_{\Gamma_R}(\frac{\partial v}{\partial \nu} + \alpha v)^2}{\|\mathbf{x}\|^2}\right\} \leq CN(1+w^2) \leq 2CN. \tag{6.7}$$

The challenging work is to estimate the lower bound of $\lambda_{min}(\mathbf{B})$ in (6.4).

For simplicity, we assume $R \neq 1$, and $\Gamma_D \neq \emptyset$ (or $\Gamma_D = \emptyset$ but $\alpha > 0$). Hence the matrix $\mathbf{B}$ is nonsingular, and then $\lambda_{min}(\mathbf{B}) > 0$. Denote an inside disk $S_0 \leq S$ with the boundary,

$$\begin{aligned} S_0 &= \{(r,\theta)|\ r \leq r_{min},\ 0 \leq \theta \leq 2\pi\}\,, \\ \Gamma_0 &= \{(r,\theta)|\ r = r_{min},\ 0 \leq \theta \leq 2\pi\}\,. \end{aligned} \tag{6.8}$$

We have the following lemma.

Lemma 6.1 *Suppose that there exists a positive constant μ independent of N such that*

$$|v|_{1,\Gamma_D} \le CN^{\mu} \|v\|_{0,\Gamma_D}, \ v \in V_N, \tag{6.9}$$

where V_N is the set of (2.9), and C is a constant independent of N. Then there exists the bound

$$I_0(v) \ge c_0 \left\{N^{-\mu}, w^2\right\} \|v\|^2_{0,\Gamma_0}, \tag{6.10}$$

where $c_0(> 0)$ is a constant independent of N.

Proof : In the following, C is always a constant independent of N, but the values of C may be different in different texts. For the harmonic functions $v \in V_N$ in S, we have from Odden and Reddy [45]

$$\|v\|_{1,S} \le C\left\{\|v\|_{\frac{1}{2},\Gamma_D} + \left\|\frac{\partial v}{\partial \nu} + \alpha v\right\|_{-\frac{1}{2},\Gamma_R}\right\}. \tag{6.11}$$

Since (6.9) we have

$$\|v\|_{\frac{1}{2},\Gamma_D} \le C\left\{\|v\|_{1,\Gamma_D}\|v\|_{0,\Gamma_D}\right\}^{\frac{1}{2}} \le CN^{\frac{\mu}{2}}\|v\|_{0,\Gamma_D} \tag{6.12}$$

and

$$\left\|\frac{\partial v}{\partial \nu} + \alpha v\right\|_{-\frac{1}{2},\Gamma_R} \le C\left\|\frac{\partial v}{\partial \nu} + \alpha v\right\|_{0,\Gamma_R}. \tag{6.13}$$

Hence we obtain from (6.11) – (6.13)

$$\begin{aligned}\|v\|^2_{1,S} &\le C\left\{\|v\|^2_{\frac{1}{2},\Gamma_D} + \left\|\frac{\partial v}{\partial \nu} + \alpha v\right\|^2_{-\frac{1}{2},\Gamma_R}\right\} \\ &\le C\left\{N^{\mu}\|v\|^2_{0,\Gamma_D} + \left\|\frac{\partial v}{\partial \nu} + \alpha v\right\|^2_{0,\Gamma_R}\right\} \\ &\le C\max\left\{N^{\mu}, \frac{1}{w^2}\right\} I_0(v),\end{aligned} \tag{6.14}$$

where $I_0(v)$ is given in (6.2). This gives

$$I_0(v) \geq c_0 \min\left\{N^{-\mu}, w^2\right\} \|v\|_{1,S}^2 \,, \tag{6.15}$$

where $c_0(>0)$ is a constant independent of N. Moreover, since $S_0 \subset S$, and the imbedding theorem

$$\|v\|_{0,\Gamma_0} = \|v\|_{0,\partial S_0} \leq C\|v\|_{1,S_0},$$

we have

$$\|v\|_{1,S} \geq \|v\|_{1,S_0} \geq c_0\|v\|_{0,\Gamma_0}. \tag{6.16}$$

Combining (6.15) and (6.16) yields the desired result (6.10). This completes the proof of Lemma 6.1. ∎

We have the following theorem.

Theorem 6.1 *Let $R \neq 1$ and (6.9) hold. There exists the bound,*

$$\lambda_{min}(\mathbf{B}) \geq c_0 \min\left\{N^{-\mu}, w^2\right\} \frac{1}{N}\left(\frac{r_{min}}{R}\right)^{2N-2}, \tag{6.17}$$

where $c_0\,(>0)$ is a constant independent of N.

Proof : We have

$$\|v\|_{0,\Gamma_0}^2 = \int_{\Gamma_0} v^2 = \int_{\Gamma_0}\left(\sum_{i=1}^{N} c_i\phi_i(P)\right)^2$$
$$= \sum_{i,j=1}^{N} c_i\, c_j \int_{\Gamma_0} \phi_i(P)\phi_j(P)\ , \quad P \in \Gamma_0\,. \tag{6.18}$$

Since $R > r_{min}$, the fundamental functions $\phi_i(P)$ on Γ_0 belong $C^\infty[0,\pi]$, and satisfy

$$\phi_i^k(0) = \phi_i^k(2\pi), \quad i = 0, 1, \cdots$$

We have from Lemma 4.2

$$\int_{\Gamma_0} \phi_i(P)\phi_j(P) = \widehat{\int}_{\Gamma_0} \phi_i(P)\phi_j(P) = h\, r_{min} \sum_{\ell=1}^{N} \phi_i(P_\ell)\phi_j(P_\ell), \tag{6.19}$$

where $h = \frac{2\pi}{N}$, and

$$P_\ell = (r_{min}\, \cos \ell h\ ,\ r_{min}\, \sin \ell h)\,, \quad \ell = 1, 2, \cdots, N.$$

Combining (6.18) and (6.19) gives

$$\begin{aligned} \|v\|^2_{0,\Gamma_0} &= h\, r_{min} \sum_{i,j=1}^{N} c_i c_j \sum_{\ell=1}^{N} \phi_i(P_\ell)\phi_j(P_\ell) = h\, r_{min} \sum_{\ell=1}^{N} \sum_{i,j=1}^{N} c_i c_j\, \phi_i(P_\ell)\phi_j(P_\ell) \\ &= h\, r_{min} \sum_{\ell=1}^{N} \left(\sum_{i=1}^{N} c_i \phi_i(P_\ell) \right)^2 = h\, r_{min} \mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x}, \end{aligned} \tag{6.20}$$

where $\mathbf{A}$ is the circulant matrix (4.8) with $\rho = r_{min}$. From Theorem 4.1,

$$\frac{\mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \geq \lambda^2_{N-1}(\mathbf{A}) \geq \frac{1}{4} \left(\frac{r_{min}}{R} \right)^{2N-2}. \tag{6.21}$$

Form Lemma 6.1, (6.4), (6.20) and (6.21), we obtain

$$\begin{aligned} \lambda_{min}(\mathbf{B}) &\geq c_0 \left\{ N^{-\mu}, w^2 \right\} \frac{\|v\|^2_{0,\Gamma_0}}{\|\mathbf{x}\|^2} \geq c_0\, h\, r_{min} \left\{ N^{-\mu}, w^2 \right\} \frac{\mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x}}{\|\mathbf{x}\|^2} \\ &\geq \bar{c}_0 \left\{ N^{-\mu}, w^2 \right\} \frac{1}{N} \left(\frac{r_{min}}{R} \right)^{2N-2}, \end{aligned} \tag{6.22}$$

where $\bar{c}_0$ (> 0) is also a constant indepenpent of N. This is the desired result (6.17), and completes the proof of Theorem 6.1. ∎

From Theorem 6.1 and (6.7), we obtain the following corollary.

Corollary 6.1 *Let $R > r_{min}$, $R \neq 1$ and (6.9) hold. Choose $w = N^{-\frac{\mu}{2}}$. There exists the bound*

$$\text{Cond}(\mathbf{B}) \leq C\, N^{2+\mu} \left(\frac{R}{r_{min}} \right)^{2N}. \tag{6.23}$$

Remark 6.1 For the pure Robin problem in the boundary domain S. We do not need the inverse inequality (6.9). By following the above arguments, we have the better bound

$$\text{Cond}(\mathbf{B}) \leq C\, N^2 \left(\frac{R}{r_{min}} \right)^{2N}. \tag{6.24}$$

6.2 The Collocation Trefftz Methods

In computation, we use (2.17) and (2.18) with (2.19), and obtain the overdetermined system

$$\mathbf{F}\mathbf{x} = \mathbf{d}, \tag{6.25}$$

where $\mathbf{x} = (c_1, \cdots, c_N)^T, \quad \mathbf{d} \in R^M$, and $\mathbf{F} \in R^{M \times N}$. It is easy to see that

$$\frac{1}{2}\mathbf{x}^T\mathbf{F}^T\mathbf{F}\mathbf{x} = \hat{I}_0(v), \tag{6.26}$$

where

$$\hat{I}_0(v) = \widehat{\int}_{\Gamma_D}(v-f)^2 + w^2\widehat{\int}_{\Gamma_R}(\frac{\partial v}{\partial \nu} + \alpha v)^2. \tag{6.27}$$

Denote the discrete norms

$$\overline{\|v\|}_{0,\Gamma_D} = (\widehat{\int}_{\Gamma_D} v^2)^{\frac{1}{2}}, \quad \overline{\|v\|}_{0,\Gamma_R} = (\widehat{\int}_{\Gamma_R} v^2)^{\frac{1}{2}}, \tag{6.28}$$

where $\widehat{\int}_{\Gamma_D}$ and $\widehat{\int}_{\Gamma_R}$ are the numerical approximations of $\int_{\Gamma_D}$ and $\int_{\Gamma_R}$ in (2.10) by the central or the Gaussian rule. In Lu et al. [40], the following equivalent norms hold

$$\overline{\|v\|}_{0,\Gamma_D} \asymp \|v\|_{0,\Gamma_D}, \quad \overline{\|v\|}_{0,\Gamma_R} \asymp \|v\|_{0,\Gamma_R}, \tag{6.29}$$

by rough integral approximation, e.g., the central rule. In (6.29), the notation $a \asymp b$ or $a \asymp O(b), \quad b > 0$ means that there exist two positive constants c_1 and c_2 such that

$$c_1 b \ \leq \ a \ \leq \ c_2 b. \tag{6.30}$$

Denote the singular values σ_i of $\mathbf{F}$ in (6.25) and

$$\sigma_{max} = \max_i \sigma_i\,, \quad \sigma_{min} = \min_i \sigma_i \ > 0.$$

The MFS is just the CTM using FS. We have the following theorem.

Theorem 6.2 *Let $R \neq 1$, (6.9) and (6.29) hold. For the MFS in (2.17) and (2.18) with (2.19), there exist the bounds*

$$\sigma_{max}(\mathbf{F}) \leq C\, N^{\frac{1}{2}}, \tag{6.31}$$

$$\sigma_{min}(\mathbf{F}) \geq \bar{c}_0 \left\{ L^{-\frac{\mu}{2}}, w \right\} \frac{1}{\sqrt{N}} \left(\frac{r_{min}}{R} \right)^{N-1}. \tag{6.32}$$

Proof : From (6.29), we have

$$\hat{I}_0(v) \asymp I_0(v), \tag{6.33}$$

where $\hat{I}_0(v)$ and $I_0(v)$ are given in (6.27) and (6.2) respectively. From (6.7) and (6.33) we have

$$\begin{aligned} \lambda_{max}(\mathbf{F}^T\mathbf{F}) &= \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T\mathbf{F}^T\mathbf{F}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \max_{\substack{x\neq 0\\ v\in V_N}} \frac{2\,\hat{I}_0(v)}{\|\mathbf{x}\|^2} \\ &\leq C \max_{\substack{x\neq 0\\ v\in V_N}} \frac{I_0(v)}{\|\mathbf{x}\|^2} \leq C\lambda_{max}(\mathbf{B}) \leq \bar{C}\, N, \end{aligned} \tag{6.34}$$

where $\bar{C}$ is a constant independent of N. Next, from Theorem 6.1 and (6.32) we have

$$\begin{aligned} \lambda_{min}(\mathbf{F}^T\mathbf{F}) &= \min_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T\mathbf{F}^T\mathbf{F}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \min_{\substack{x\neq 0\\ v\in V_N}} \frac{2\,\hat{I}_0(v)}{\|\mathbf{x}\|} \\ &\geq c_0 \min_{\substack{x\neq 0\\ v\in V_N}} \frac{I_0(v)}{\|\mathbf{x}\|} \geq \bar{c}_0 \min\left\{ N^{-\mu}, w^2 \right\} \frac{1}{N} \left(\frac{r_{min}}{R} \right)^{2N-2}. \end{aligned} \tag{6.35}$$

The desired results (6.31) and (6.32) follow from (6.34), (6.35) and the following equalities

$$\sigma_{max}(\mathbf{F}) = \left\{ \lambda_{max}(\mathbf{F}^T\mathbf{F}) \right\}^{\frac{1}{2}}, \qquad \sigma_{min}(\mathbf{F}) = \left\{ \lambda_{min}(\mathbf{F}^T\mathbf{F}) \right\}^{\frac{1}{2}}.$$

This completes the proof of Theorem 6.2. ∎

6.3 The Inverse Inequality (6.9)

The equation (6.9) is an important issue in our stability analysis for the Dirichlet boundary condition (i.e., $\Gamma_D \neq \emptyset$). In this subsection, we will confirm the validation of (6.9), and show $\mu \in [1, 2]$ for some simple cases.

In Figure 1, denote

$$S_{min} \subset S \subset S_{max}, \tag{6.36}$$

where

$$\begin{aligned} S_{min} &= \{(r,\theta)|r = r_{min},\ 0 \le \theta \le 2\pi\}, \\ S_{max} &= \{(r,\theta)|r = r_{max},\ 0 \le \theta \le 2\pi\}. \end{aligned} \tag{6.37}$$

Denote the trigonometric functions[2]

$$P_n = P_n(r,\theta) = \sum_{i=1}^{n} r^i(\alpha_i \cos i\theta + \beta_i \sin i\theta), \tag{6.38}$$

where α_i and β_i are coefficients, and the fundamental solutions

$$v_N = \sum_{i=1}^{N} c_i \ln |\overline{PQ_i}|, \tag{6.39}$$

where $P \in (S \cup \partial S)$, and

$$Q_i = (R \cos ih, R \sin ih), \tag{6.40}$$

$R > r_{max}$ and $h = \frac{2\pi}{N}$. We cite the results of Bogomolny [4], p. 655 as a lemma.

Lemma 6.2 *Let (6.36) hold and* Γ *be sufficient smooth. When* $q \ge 0$ *and*

$$2^{2q+1} \left(\frac{R}{r_{max}}\right)^{-2N} \ < \ 1 \tag{6.41}$$

[2]Strictly speaking, we should consider the following trigonometric functions

$$P_n(r,\theta) = \alpha_0 + \sum_{i=1}^{n} r^i(\alpha_i \cos i\theta + \beta_i \sin i\theta)$$

with the leading constant α_0. From (5.30), we may replace (6.39) by

$$v_N = \sum_{i=1}^{N} c^* \ln |\overline{PQ_i}|,$$

where $c_i^* = c_i + \frac{\alpha_0}{N \ln R}$ and $R \ne 1$. Hence we simply choose (6.38) for simple discussions (see [4]).

there exists the bound,

$$\|P_n(r,\theta) - v_N\|_{H^q(\Gamma)} \leq C\, N^q \left(\frac{R}{r_{max}}\right)^{2n-N} \left(\frac{r_{max}}{r_{min}}\right)^n \sqrt{n}\, \|P_n(r,\theta)\|_{0,\Gamma}, \qquad (6.42)$$

where $P_n(r,\theta)$ and v_n are given in (6.38) and (6.39), respectively.

Choose $q = 1$, from (6.41) we have

$$N > \frac{1.5 \ln 2}{\ln\left(\frac{R}{r_{max}}\right)}. \qquad (6.43)$$

Let N be given, and choose

$$\left(\frac{R}{r_{max}}\right)^{2n-N} \left(\frac{r_{max}}{r_{min}}\right)^n \asymp N^{-t}\,. \qquad (6.44)$$

The equation (6.42) leads to

$$\|P_n(r,\theta) - v_N\|_{1,\Gamma} \leq C N^{1-t}\sqrt{n}\|P_n(r,\theta)\|_{0,\Gamma}. \qquad (6.45)$$

From (6.44), we have

$$\left(\frac{R^2}{r_{max}\, r_{min}}\right)^n \asymp \left(\frac{R}{r_{max}}\right)^N N^{-t},$$

to give

$$n \approx \frac{N \ln\left(\frac{R}{r_{max}}\right) - t \ln N}{\ln\left(\frac{R^2}{r_{max}\, r_{min}}\right)}. \qquad (6.46)$$

In particular, choose $t = 3.5$, we have from (6.45)

$$\|P_n(r,\theta) - v_N\|_{1,\Gamma} \leq C\, N^{-2}\|P_n(r,\theta)\|_{0,\Gamma}\,. \qquad (6.47)$$

We write this result as a corollary.

Corollary 6.2 *Let (6.36) hold and* Γ *be sufficient smooth. For given* N *satisfying (6.43), when choose*

$$n \approx \frac{N \ln\left(\frac{R}{r_{max}}\right) - 3.5 \ln N}{\ln\left(\frac{R^2}{r_{max}\, r_{min}}\right)}, \tag{6.48}$$

the bound (6.47) holds.

The equation (6.46) implies $n \asymp N$. We have the following theorem.

Theorem 6.3 *Let all conditions in Corollary 6.2 hold. Also assume that for* $P_n(r, \theta)$ *in (6.38),*

$$\|P_n(r, \theta)\|_{1,\Gamma} \le c n^{\mu} \|P_n(r, \theta)\|_{0,\Gamma}\,, \tag{6.49}$$

where $\mu(> 0)$ *is a constant independent of* n. *Then there exists the bound*

$$\|v_N\|_{1,\Gamma} \le C\, N^{\mu} \|v_N\|_{0,\Gamma} \quad v \in V_N. \tag{6.50}$$

Proof : Denote $P_n = P_n(r, \theta)$ and choose n in (6.48). We obtain form (6.47) and (6.49)

$$\begin{aligned} \|v_N\|_{1,\Gamma} &\le \|P_n\|_{1,\Gamma} + \|P_n - v_N\|_{1,\Gamma} \\ &\le C\left\{ n^{\mu} \|P_n\|_{0,\Gamma} + \frac{1}{N^2} \|P_n\|_{0,\Gamma} \right\} \\ &\le \bar{C} N^{\mu} \|P_n\|_{0,\Gamma}, \end{aligned} \tag{6.51}$$

where $\bar{C}$ is a constant independent of N. On the other hand, from Corollary 6.2,

$$\begin{aligned} \|P_n\|_{0,\Gamma} &\le \|v_N\|_{0,\Gamma} + \|v_N - P_n\|_{0,\Gamma} \le \|v_N\|_{0,\Gamma} + \|v_n - P_n\|_{1,\Gamma} \\ &\le \|v_N\|_{0,\Gamma} + C \frac{1}{N^2} \|P_n\|_{0,\Gamma}. \end{aligned} \tag{6.52}$$

This gives

$$\|P_n\|_{0,\Gamma} \le \frac{1}{\left(1 - \frac{C}{N^2}\right) \|v_N\|_{0,\Gamma}} \le \bar{C} \|v_N\|_{0,\Gamma}. \tag{6.53}$$

Combining (6.51) and (6.53) yields

$$\|v_N\|_{1,\Gamma} \le C\, N^{\mu} \|v_N\|_{0,\Gamma}. \tag{6.54}$$

This completes the proof of theorem 6.3. ∎

Below, let us discuss (6.49) for three cases.

Case I. Consider the circle

$$\Gamma = \ell_\rho = \{(r,\theta)|\ r=\rho,\ 0 \le \theta \le 2\pi\}.$$

From the orthogonality of trigonometric functions, we have

$$\begin{aligned} \|P_n(r,\theta)\|^2_{1,\Gamma} &= \int_0^{2\pi} P_n^2(r,\theta)\rho\, d\theta = \pi \sum_{i=1}^{n} \rho^{2i} i^2 (\alpha_i^2 + \beta_i^2) \\ &\le \pi n^2 \sum_{i=1}^{n} \rho^{2i} (\alpha_i^2 + \beta_i^2) = n^2\, \|P_n(r,\theta)\|^2_{0,\Gamma}. \end{aligned} \tag{6.55}$$

Hence, we obtain

$$\|P_n(r,\theta)\|_{1,\Gamma} \le n \|P_n(r,\theta)\|_{0,\Gamma}\,, \tag{6.56}$$

which is (6.49) with $\mu = 1$.

Case II. Consider $\Gamma = \bigcup_{i=1}^{L} \Gamma_i$, where Γ_i are straight lines. Let $x = r\cos\theta$ and $y = r\sin\theta$. We have

$$P_n(r,\theta) = \sum_{i,j=0}^{n} a_{ij} x^i y^j = T_{2n}(s), \quad s \in \Gamma_i, \tag{6.57}$$

where $T_{2n}(s)$ is the polynomials of order $2n$ with respect to s on Γ_i. From Li [34], p. 163, we have

$$\|P_n(r,\theta)\|_{1,\Gamma_i} = \|T_{2n}(s)\|_{1,\Gamma_i} \le C(2n)^2 \|T_{2n}(s)\|_{0,\Gamma_i}. \tag{6.58}$$

Hence, we have

$$\begin{aligned}
\|P_n(x)\|_{1,\Gamma}^2 &= \sum_{i=1}^{L} \|P_n(r,\theta)\|_{1,\Gamma_i}^2 = \sum_{i=1}^{L} \|T_{2n}(s)\|_{1,\Gamma_i}^2 \\
&\le Cn^4 \sum_{i=1}^{L} \|T_{2n}(s)\|_{0,\Gamma_i}^2 \\
&= Cn^4 \sum_{i=1}^{L} \|P_n(r,\theta)\|_{0,\Gamma_i}^2 = Cn^4 \|P_n(r,\theta)\|_{0,\Gamma}^2. \qquad (6.59)
\end{aligned}$$

Then there exists the bound

$$\|P_n(x)\|_{1,\Gamma} \le Cn^2 \|P_n(x)\|_{0,\Gamma}, \qquad (6.60)$$

which is (6.49) with $\mu = 2$.

Case III. Let $\Gamma = \bigcup_{i=1}^{L} \Gamma_i$, where $Meas\,(\Gamma_i) > 0$. Suppose that Γ_i can be denoted by the parmetric polynomials

$$\Gamma_i = \Gamma_i(s) = \{(x,y) | x = q_i(s),\ y = t_i(s)\}, \qquad (6.61)$$

and

$$q_i(s) = \sum_{k=0}^{n_i} \alpha_{ik} s^k, \quad t_i(s) = \sum_{k=0}^{m_i} \beta_{ik} s^k, \qquad (6.62)$$

where α_{ik} and β_{ik} are the coefficients, and n_i and m_i are non-negative bounded integers. Denote

$$n_{max} = \max_i \{n_i, m_i\}. \qquad (6.63)$$

We have the following proposition.

Proposition 6.1 *Let Γ be given in Case III. Suppose that n_{max} is bounded. There exists the bound,*

$$\|P_n(r,\theta)\|_{1,\Gamma} \le C\, n^2 \|P_n(r,\theta)\|_{0,\Gamma}. \qquad (6.64)$$

From (6.57), we have for Γ_ℓ,

$$P_n(r,\theta) = \sum_{i,j=0}^{n} a_{ij}x^i y^j = \sum_{i,j=0}^{n} a_{ij}(q_\ell(s))^i (t_\ell(s))^j = T_{p_\ell}(s), \quad s \in \Gamma_\ell, \tag{6.65}$$

where $T_{p_\ell}(s)$ are the polynomials of order

$$p_\ell \le 2n \, \max\{n_\ell, m_\ell\} \le 2n \, n_{max}. \tag{6.66}$$

Then we have from [34],

$$\begin{aligned} \|P_n(r,\theta)\|^2_{1,\Gamma} &\le \sum_{\ell=1}^{L} \|P_n(r,\theta)\|^2_{1,\Gamma_\ell} = \sum_{\ell=1}^{L} \|T_{p_\ell}(s)\|^2_{1,\Gamma_\ell} \\ &\le C(2n)^4 n^4_{max} \sum_{i=1}^{L} \|T_{p_\ell}(s)\|^2_{0,\Gamma_\ell} \\ &\le \bar{C} n^4 \sum_{i=1}^{L} \|P_n(r,\theta)\|^2_{0,\Gamma_\ell} = \bar{C} n^4 \|P_n(r,\theta)\|^2_{0,\Gamma}, \end{aligned} \tag{6.67}$$

where $\bar{C}$ is a constant independent of N. The desired result (6.64) is obtained from (6.67), and the proof of Proposition 6.1 is completed. ∎

When $q_i(s)$ and $t_i(s)$ are linear functions, Proposition 6.1 leads to (6.60) in Case II. From Theorem 6.2 we have the following corollary.

Corollary 6.3 *Let the conditions in Theorem 6.2 hold. Also let* $\mu \in [1,2]$ *and choose* $w = L^{-\frac{\mu}{2}}$. *There exists the bound*

$$\text{Cond}(\mathbf{F}) = \frac{\sigma_{max}(\mathbf{F})}{\sigma_{min}(\mathbf{F})} \le C \, N^{(1+\frac{\mu}{2})} \left(\frac{R}{r_{min}}\right)^{N-1}. \tag{6.68}$$

The equation (6.68) indicates the exponential rates of Cond by MFS, to coincide with the conclusions in Sections 4 and 5.

6.4 Method of Fundamental Solutions

Also consider Motz's problem (2.1) – (2.3) in regular S, and choose the fundamental solutions (FS),

$$u_N = \sum_{i=1}^{N} c_i \, \phi_i(P), \tag{6.69}$$

where c_i are the coefficients, and

$$\phi_i(P) = \ln|P Q_i|, \quad P \in (S \cup \partial S). \tag{6.70}$$

In (6.70), the source points Q_i are located uniformly on a large circule at the origin $(0, \frac{1}{2})$, see Figure 5

$$Q_i = \{(x_i, y_i) \,|\, x_i = R\cos i\Delta\theta, \, y_i = R\sin i\Delta\theta\}, \tag{6.71}$$

where $\Delta\theta = \frac{2\pi}{N}$, $R > \frac{\sqrt{5}}{2}$, and (r, θ) are the polar coordinates with the origin $(0, \frac{1}{2})$ in Figure 5. Define by V_N the functions (6.69), and the energy

$$\hat{I}(v) = \widehat{\int}_{\overline{AB}} (v-500)^2 + \widehat{\int}_{\overline{OD}} v^2 + \widehat{\int}_{\overline{BC}\cup\overline{CD}\cup\overline{OA}} v_\nu^2. \tag{6.72}$$

The CTM using FS is expressed by

$$\hat{I}(u_N) = \min_{v\in V_N} \hat{I}(v). \tag{6.73}$$

Since functions (6.69) satisfy (2.1) already, we way establish the boundary

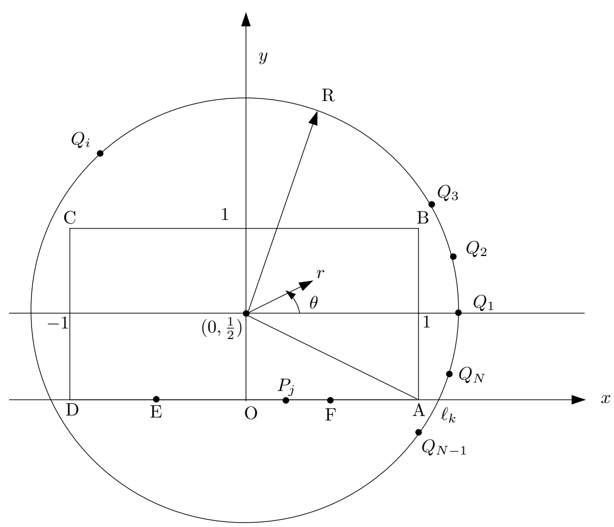

Figure 3: Motz's problem with its source points.

collocation equations directly from (2.2) and (2.3),

$$\alpha_j \sum_{i=1}^{N} c_i \phi_i(P_j) = 0, \qquad P_j \in \overline{OD}, \tag{6.74}$$

$$\alpha_j \sum_{i=1}^{N} c_i \phi_i(P_j) = \alpha_j \, 500, \qquad P_j \in \overline{AB}, \tag{6.75}$$

$$\alpha_j \, w \left\{ \sum_{i=1}^{N} c_i \frac{\partial}{\partial \nu} \phi_i(P_j) \right\} = 0, \quad P_j \in \overline{BC} \cup \overline{CD} \cup \overline{OA}, \tag{6.76}$$

where w is the weight as $\frac{1}{N}$, α_j are the weights in quadrature with $\alpha_j = O(\sqrt{h})$, and h is the maximal meshspacing of collocation nodes P_j. Since there exists a singularity at O in Figure 5 due to the intersection of the Dirichlet and Neumann boundary conditions, we adopt the local refinements of P_j near O (see Figure 5),

$$|\overline{OP_j}| = (jh)^q , \quad j = 1, 2, \cdots \tag{6.77}$$

where $q \geq 1$. Once the approximate solutions (6.69) have been obtained, the leading coefficients can be evaluated by

$$D_\ell = \frac{2}{\pi r^{\ell+\frac{1}{2}}} \int_0^\pi u_N(r,\theta) \cos(\ell + \frac{1}{2})\theta \, d\theta, \quad \ell = 0, 1, 2, 3. \tag{6.78}$$

The leading coefficient D_0 indicates the intensity factor, which is important in both theory and application.

For smooth solutions, we cite a lemma.

Lemma 6.3 *Suppose that there exists a positive number* ($\mu > 0$) *such that*

$$|v|_{1,\Gamma_D} \leq CN^\mu \|v\|_{0,\Gamma_D} , \quad \forall v \in V_N . \tag{6.79}$$

Then there exist the bounds,

$$\sigma_{max} \leq C\sqrt{N} , \tag{6.80}$$

$$\sigma_{min} \geq c_0 \min \left\{ N^{-\frac{\mu}{2}} , \, w \right\} \frac{1}{\sqrt{N}} \left(\frac{r_{min}}{R} \right)^N , \tag{6.81}$$

where C and c_0 (> 0) are two constant independent of N, and $r_{min} = \min_S r$.

We have the following corollary from Lemma 6.3.

Corollary 6.4 *Let (6.79) hold. There exist the bounds,*

$$\text{Cond} \leq CN \max \left\{ N^{\frac{\mu}{2}}, \frac{1}{w} \right\} \left(\frac{R}{r_{min}} \right)^N , \tag{6.82}$$

$$\text{Cond\_eff} \leq C \frac{\sqrt{N}}{\|\mathbf{x}\|} \max \left\{ N^{\frac{\mu}{2}}, \frac{1}{w} \right\} \left(\frac{R}{r_{min}} \right)^N . \tag{6.83}$$

When choose $w = N^{-\frac{\mu}{2}}$, there exist the bounds,

$$\text{Cond} \leq CN^{1+\frac{\mu}{2}} \left(\frac{R}{r_{min}} \right)^N . \tag{6.84}$$

$$\text{Cond\_eff} \leq C \frac{N^{\frac{1+\mu}{2}}}{\|\mathbf{x}\|} \left(\frac{R}{r_{min}} \right)^N . \tag{6.85}$$

Proof : The equation (6.82) follows from (6.80) and (6.81) directly. Next we show (6.83). Since $b_j = \alpha_j 500$, $P_j \in \overline{AB}$, we have

$$\|\mathbf{b}\| = \sum_j (\alpha_j 500)^2 = 500^2 \int_{\overline{AB}} d\ell = 500^2 \asymp O(1).$$

Then we have

$$\text{Cond\_eff} = \frac{\|\mathbf{b}\|}{\sigma_{min}\|\mathbf{x}\|} \le C\frac{1}{\sigma_{min}\|\mathbf{x}\|} = C\frac{\sqrt{N}}{\|\mathbf{x}\|}\max\left\{N^{\frac{\mu}{2}}, \frac{1}{w}\right\}\left(\frac{R}{r_{min}}\right)^N.$$

This is the desired result (6.83). The equations (6.84) and (6.85) are easily obtained from (6.82) and (6.83). This completes the proof of Corollary 6.4. ∎

Define the boundary errors by MFS

$$\|\varepsilon\|_B = \left\{\int_{\overline{AB}\cup\overline{DE}} \varepsilon^2 + \int_{\overline{BC}\cup\overline{CD}\cup\overline{EA}} \left(\frac{\partial\varepsilon}{\partial\nu}\right)^2\right\}^{\frac{1}{2}}, \tag{6.86}$$

where $\varepsilon = u - u_N$ and $\overline{AB}$, $\overline{DE}$, etc. are given in Figure 5. The radius $R = \sqrt{3}$ in (6.71) is chosen, based on the numerical experiments. For Motz's problem, the errors and condition numbers are listed in Tables 10 – 12, and the coefficients in (6.70) in Table 13. The uniform collocation points with $h = \frac{1}{M}$ are used on ∂S, except those on $\overline{OE} \cup \overline{OF}$ in Figure 5, where the collocation points P_j are given by (6.77). In computation, we choose $|\overline{OF}| = |\overline{OE}| = \frac{1}{8}$ and retain the invariant M of collocation points on $\overline{OA}$ and $\overline{OD}$. From Table 10, when the uniform collocation nodes are chosen, the FS solutions are poor. From Table 11, as $q = 4$ in (6.77) is a good choice of local refinements. In Table 12, the errors and condition numbers are provided for $R = \sqrt{3}$ and $q = 4$, and the following empirical rates can be observed:

$$\|\varepsilon\|_B = O(0.74^N), \qquad \left|\frac{\Delta D_0}{D_0}\right| = O(0.97^N), \tag{6.87}$$

$$\sigma_{max} = O(1), \qquad \sigma_{min} = O(0.487^N), \tag{6.88}$$

$$\text{Cond} = O(2.05^N), \quad \text{Cond\_eff} = O(1.37^N), \tag{6.89}$$

$$\|\mathbf{x}\| = O(1.47^N).$$

Evidently, the Cond_eff is much smaller than Cond. The norm $\|\mathbf{x}\|$ is large. Note that some coefficients c_i in Table 12 are huge and highly oscillating. Hence, the subtraction cancellation occurs in the final harmonic solution (6.69).

The behavior of empirical rates for Motz's problem and the Smooth problem is similar. For the smooth problem, the solution is more accurate, but the ill-conditioning is worse. Also, the norm $\|\mathbf{x}\|$ is also large. There exists the empirical relation from (7.15) – (7.17)

$$\text{Cond} \approx \text{Cond\_eff}\, \|\mathbf{x}\|\,. \tag{6.90}$$

To reduce the severe ill-conditioning of MFS, where σ_{min} is infinitesimal, the truncated singular value decomposition and the Tirhonov regularization can be solicited, details are reported in Part II.

The solution by the singular value decomposition (SVD) is expressed by

$$\mathbf{x} = \sum_{i=1}^{n} \frac{\beta_i}{\sigma_i} \mathbf{v}_i\,, \tag{6.91}$$

where $\beta_i = \mathbf{u}_i^T \mathbf{b}$, σ_i are the singular values of $\mathbf{F}$, and $\mathbf{u}_i$ and $\mathbf{v}_i$ are the vectors of $\mathbf{U}$ and $\mathbf{V}$ respectively

$$\mathbf{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_m)^T\,, \quad \mathbf{V} = (\mathbf{v}_1, \cdots, \mathbf{v}_m)^T\,.$$

When $\sigma_{min} = \sigma_n$ is infinitesmal, we may use the truncated SVD, to obtain the solution

$$\mathbf{x}_k = \sum_{i=1}^{k} \frac{\beta_i}{\sigma_i} \mathbf{v}_i\,, \quad k \le n\,, \tag{6.92}$$

to replace (6.92). Also, choose a parameter $\lambda \in [\sigma_{min}, \sqrt{\sigma_{min}\sigma_{max}}]$, and use the Tikhonov regularization to obtain

$$\mathbf{x}_\lambda = \sum_{i=1}^{n} \left(\frac{\sigma_i^2}{\sigma_i^2 + \lambda^2} \right) \frac{\beta_i \mathbf{v}_i}{\sigma_i}\,. \tag{6.93}$$

Detail of TSVD and TR are given in Part II.

Take $N = 84$ and $M = 60$ in Table 12 for example. The errors and condition numbers are listed in Tables 14 and 15. We can see that the norm $\|\mathbf{x}_k\|$ decreases dramatically from $O(10^{10})$ to $O(10^8)$, while the errors $\|\varepsilon\|_B$ increase only by a factor of 10. Also the Cond and Cond_eff are reduced correspondingly.

7 Numerical Experiments

7.1 MFS for Motz's problem by adding singular functions

Consider Motz's problem (see Figure 4)

$$\Delta u = 0, \tag{7.1}$$
$$u = 500 \quad \text{on } \overline{AB}, \qquad u = 0 \quad \text{on } \overline{DE}, \tag{7.2}$$
$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \overline{BC} \cup \overline{CD} \cup \overline{EA}. \tag{7.3}$$

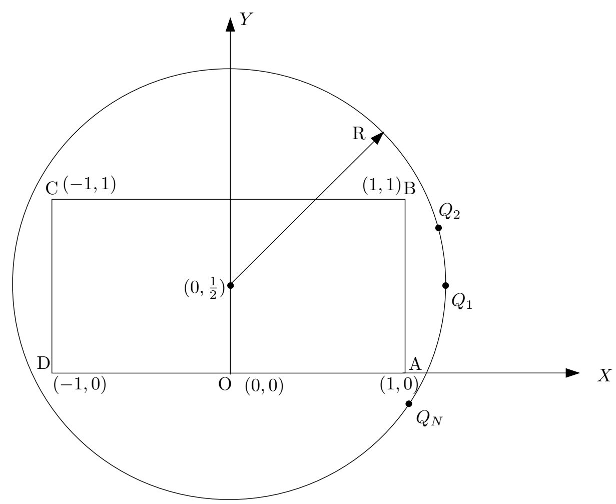

Figure 4: Motz's problem with the source circle ℓ_R.

There exists a singularity at E due to the intersection of the Dirichlet and the Neumann boundary condition. Since the fundamental functions (2.8) are smooth on ∂S, the local refinements of collocation nodes P_i near E may be used for Motz's problem; the results will report elsewhere. In this thesis, we will add the singular functions into (2.9), to obtain

$$u_N = \sum_{i=0}^{L} D_i r^{i+\frac{1}{2}} \cos(i + \frac{1}{2})\theta + \sum_{i=1}^{N} c_i \phi_i(P)\,, \tag{7.4}$$

where the singular particular solutions

$$\sum_{i=1}^{L} D_i r^{i+\frac{1}{2}} \cos(i + \frac{1}{2})\theta \tag{7.5}$$

are given in Li [34]. To deal with the singularity, the admissible functions as in (7.4) are also in Alres and Leitao in [1] for an enrich MFS domain decomposition technique. Let M denote the number of collocation points along $\overline{AB}$. Then the total number of collocation points on ∂S in Figure 4 is $m = 6M$. The number of unknown coefficients in (7.4) is $L + N + 1$. In our computation we choose

$$m = 6M \ \geq \ L + N + 1 = n\,. \tag{7.6}$$

From Lu et al. [40], using the Gaussian rule is beneficial to the accuracy of the singular coefficient D_i in (7.5). Then we choose the Gaussian rule of six nodes, and M is multiples of 6.

Define the boundary errors

$$\|\varepsilon\|_B \ = \left\{ \widehat{\int}_{\overline{AB}\cup\overline{DE}} \varepsilon^2 + \widehat{\int}_{\overline{BC}\cup\overline{CD}\cup\overline{EA}} \left(\frac{\partial \varepsilon}{\partial \nu}\right)^2 \right\}^{\frac{1}{2}}, \tag{7.7}$$

where $\varepsilon = u - \tilde{u}_N$. We use the SVD to solve (2.17) and (2.18). Choose $R = \sqrt{3}$ and $L = 3$, the errors and condition numbers are listed in Table 1. Table 2 shows that $L = 3$ is for better accuracy, and Table 3 shows that $\frac{M}{N} = 1.4$ is a good choice. From Table 1, we can see the empirical rates for $N \leq 70$

$$\|\varepsilon\|_B = O(0.8^N), \qquad \left|\frac{\Delta D_0}{D_0}\right| = O(0.8^N), \tag{7.8}$$

and

$$\text{Cond} = O(1.8^N), \qquad \text{Cond\_eff} = O(1.2^N). \tag{7.9}$$

From (7.8) and (7.9), both errors and condition numbers are exponential, with respect to N. The exponential rates of Cond coincide with the stability analysis of MFS in Sections 5 for the mixed boundary value problems. Note that the growth rates of Cond_eff are lower than those of Cond, to indicate that Cond_eff is a better estimate on the true stability of rounding errors, also see [35]. To reduce the condition number, Table 4 lists the results when $R \longrightarrow r_{max} = \frac{\sqrt{5}}{2}$. When R is closer to r_{max}, the errors $\|\varepsilon\|_B$ increase from $O(10^{-8})$ to $O(10^{-3})$, but the condition numbers Cond decrease from $O(10^{14})$ to $O(10^{10})$, and Cond_eff from $O(10^{10})$ to $O(10^5)$. Hence a balance between errors and ill-conditioning should be taken for practical computation.

From Table 4 we can see that the norm $\|\mathbf{x}\|$ is large, which is caused by small σ_{min}. Since the vector u_n in the orthogonal matrix $\mathbf{U}$ in (3.5) is of high frequencies, the solution

$\mathbf{x}$ of coefficients is also of high frequencies. Table 6 lists all coefficients for $N = 70$ and $M = 50$. We can see that the coefficients c_i in (7.4) obtained from the MFS are large and hight oscillating. Such a behavior of $\mathbf{x}$ is against the ideal solution in (4.46). After D_i and c_i have been obtained, the final harmonic solutions are computed by (7.4). Note that since $R - r_{max} \geq c_0 > 0, \quad |\phi_i(P)| = \Big| \ln |\overline{PQ_i}| \Big|$ is bounded. Also the harmonic solutions $|u_N| \leq 1$. Then there occurs the severe subtraction cancellation in (7.4), which is another kind of instability. On the other hand, the Cond and Cond_eff work for seeking D_i and c_i by SVD, to evaluate the ill-conditioning of rounding errors. Hence the final solutions (7.4) suffer in large instability from both rounding errors and subtraction cancellation.

To reduce the ill-conditioning of MFS, the truncated singular decomposition (TSVD) and the Tikhonov regularization (TR) can be used, see Part II. When σ_t is infinitesmall, to replace (3.7) we may choose the TSVD to obtain

$$\mathbf{x}_k = \sum_{i=1}^{k} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i \,, \quad k \leq t \,. \tag{7.10}$$

and the TR in Part II

$$\mathbf{x}_\lambda = \sum_{i=1}^{k} \left(\frac{\sigma_i^2}{\sigma_i^2 + \lambda} \right) \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i \,, \tag{7.11}$$

where $\lambda \in [\sigma_t, \sigma_{max}]$. The details are provided in Part II.

Take $N = 70$ and $M = 50$ in Table 1 for example. The errors and condition numbers are listed in Tables 6 and 7. We can see that the norm $\|\mathbf{x}_k\|$ decreases dramatically from $O(10^5)$ to $O(10^2)$, while the errors $\|\varepsilon\|_B$ increase only by a factor of 10. Also, the Cond and Cond_eff are reduced correspondingly.

To close this thesis, we assign the source points uniformly on the outside non-circular curves. We choose an outside ellipse (Figure 5)

$$\ell_{ell} = \left\{ (x, y) \,\text{satisfying}\; \frac{x^2}{(2a)^2} + \frac{\left(y - \frac{1}{2}\right)^2}{a^2} = 1 \right\}, \tag{7.12}$$

with $a > \sqrt{\frac{5}{13}}$, and an outside rectangle (Figure 6)

$$\ell_{rect} = \left\{ (x, y) \,\text{satisfying}\; -(1 + b) \leq x \leq 1 + b, \, -b \leq y - \frac{1}{2} \leq 1 + b \right\}, \tag{7.13}$$

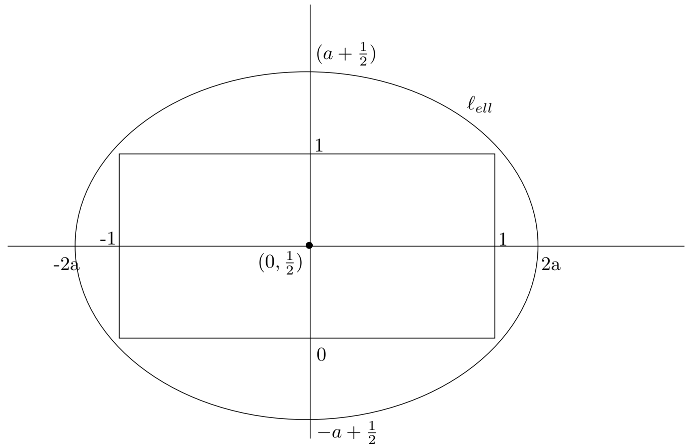

Figure 5: An outside ellipse ℓ_{ell} of S.

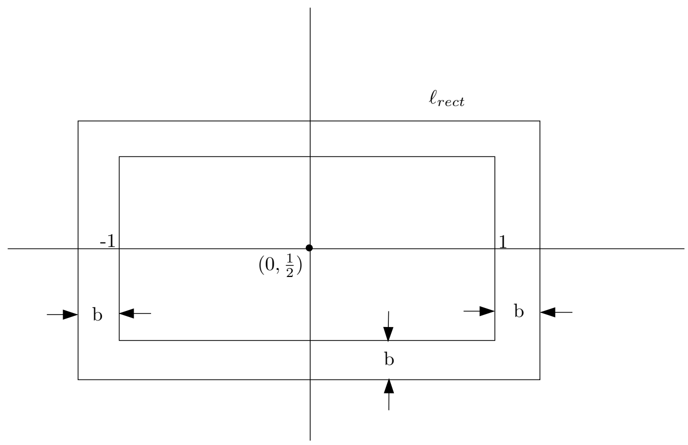

Figure 6: An outside rectangle ℓ_{rect} of S.

with $b > 0$. The source points are signed uniformly on ℓ_{ell} or on ℓ_{rect}. The collocation points are located as done before, and only the MFS is carried. A similar choice of source points as in Figure 6 can be found in Alves and Valtche [2]. The computed results are listed in Tables 8 and 9. Compared Tables 8 and 9 with Table 1 of the outside circle ℓ_R, the improvements of reducing $\|\mathbf{x}\|$ in Table 9 of ℓ_{rect} are significant, but those of ℓ_{ell} are not. Take $N = 84$ and $M = 60$ for example. From Tables 1, 8 and 9, we list the errors $\|\varepsilon\|_B$, $\|\mathbf{x}\|$, Cond and Cond_eff respectively,

$$\begin{aligned}
\|\varepsilon\|_B &= 0.6371(-9), \quad 0.201(-7) \quad \text{and} \quad 0.621(-8)\\
\|\mathbf{x}\| &= 0.1660(8), \quad 0.675(10) \quad \text{and} \quad 8938\\
\text{Cond} &= 0.2610(16), \quad 0.268(19) \quad \text{and} \quad 0.239(14)\\
\text{Cond_eff} &= 1.043(10), \quad 0.328(11) \quad \text{and} \quad 0.235(12),
\end{aligned}$$

for the rectangle ℓ_{rect}, although $\|\mathbf{x}\| = 8938$ is not large and much smaller then $\|\mathbf{x}\| = 0.1660(8)$, the improvements of both Cond and Cond_eff are not significant. For the ellipse ℓ_{ell}, all $\|\varepsilon\|_B$, $\|\mathbf{x}\|$, Cond and Cond_eff are larger than those for the circle ℓ_R. Note that the error $\|\varepsilon\| = 0.6371(-9)$ for ℓ_R is smallest. Based on the above comparisons, the circle ℓ_R is still efficient for the source points to sign in the MFS.

7.2 MFS by local refinements (6.77) of collocation nodes for Motz's problem

Replace (7.4) by

$$\mathbf{u}_N = \sum_{i=1}^{N} c_i \phi_i(p) . \tag{7.14}$$

Define the boundary errors by MFS is the same (7.7), where $\varepsilon = u - u_N$, and $\overline{AB}$, $\overline{DE}$, etc. are given in Figure 4. The radius $R = \sqrt{3}$ in (6.71) is chosen, based on the numerical experiments in Table 4. For Motz's problem, the errors and condition numbers are listed in Tables 10 – 12, and the coefficients in (6.70) in Table 13. The uniform collocation points with $h = \frac{1}{M}$ are used on ∂S, except those on $\overline{OE} \cup \overline{OF}$ in Figure 4, where the collocation points P_j are given by (6.77). In computation we choose $|\overline{OF}| = |\overline{OE}| = \frac{1}{8}$ and retain the invariant M of collocation points on $\overline{OA}$ and $\overline{OD}$. From Table 10, when the uniform collocation nodes are chosen, the FS solutions are poor. From Table 11, as $q = 4$ in (6.77) is a good choice of local refinements. In Table 12, the errors and condition numbers are provided for $R = \sqrt{3}$ and $q = 4$, and the following empirical rates can be observed:

$$\|\varepsilon\|_B = O(0.7^N), \qquad \left|\frac{\Delta D_0}{D_0}\right| = O(0.97^N), \tag{7.15}$$

$$\sigma_{max} = O(1), \qquad \sigma_{min} = O(0.4^N), \tag{7.16}$$

$$\text{Cond} = O(2.01^N), \quad \text{Cond\_eff} = O(1.37^N), \tag{7.17}$$

$$\|\mathbf{x}\| = O(1.47^N).$$

Evidently, the Cond_eff is much smaller than Cond. The norm $\|\mathbf{x}\|$ is large. Note that some coefficients c_i in Table 12 are huge and highly oscillating. Hence, the subtraction cancellation occurs in the final harmonic solution (7.14).

To compare two methods of MFS: **I** adding singular particular solutions in (7.4) in the previous subsection. **II** (7.14) using local refinements of collocation nodes near O in this subsection. We cite the main errors and conditions from Tables 1 and 12 at $N = 70$ and $M = 50$.

	Method **I**	Method **II**
$\|\varepsilon\|_B$	1.982(-8)	2.242(-7)
$\lvert\frac{\Delta D_0}{D_0}\rvert$	4.802(-8)	6.822(-3)
$\lvert\frac{\Delta D_1}{D_1}\rvert$	5.206(-6)	3.122(-2)
Cond	7.458(13)	5.758(15)
Cond_eff	6.844(9)	9.992(7)
$\|\mathbf{x}\|$	7.901(5)	4.211(9)
Tot-ill	5.41(15)	4.21(17)

Evidently, the solutions of MFS by Method **I** are more accurate, in particular, the leading coefficients are much more accurate. Such results are reasonable, become the singular particular solutions $r^{i+\frac{1}{2}}\cos(i+\frac{1}{2})\theta$ are best to match Motz's problem. However, the local refinement techniques of Method **II** may be applied to general problems of singularities, where the singular particular solutions are unknown. Next, we compare stability. Since the stability results from the effects of both rounding errors and the substraction cancellation in (7.4). It is more appropriate if we compare[3]

$$\text{Tot-ill} = \text{Cond\_eff} \times \|\mathbf{x}\|.$$

Tol-ill = 5.41(15) and 4.70(17) are obtained from Methods **I** and **II**, respectively. Hence, Method **I** has a better stability. From the huge Tot-ill, both methods suffer from severe ill-conditioning.

[3]Since Tot-ill ≈ Cond, we may simply compare the values of Cond to denote the entire stability. However, based on the traditional Cond only for rounding errors, Tot-ill-old = Cond $\times\|\mathbf{x}\|$, which is also much larger than Tot-ill.

N	28	56	70	84	112
M	20	40	50	60	80
$\|\varepsilon\|_B$	0.536(-4)	0.581(-7)	0.198(-7)	0.637(-8)	0.132(-8)
D_0	401.15985538326	401.16241731078	401.16243448162	401.16244628868	401.16245133388
$\lvert\frac{\Delta D_0}{D_0}\rvert$	0.648(-5)	0.908(-7)	0.480(-7)	0.186(-7)	0.601(-8)
D_1	87.567762254026	87.655630978127	87.655463873408	87.655806577205	87.655865753394
$\lvert\frac{\Delta D_1}{D_1}\rvert$	0.101(-2)	0.329(-5)	0.521(-5)	0.129(-5)	0.621(-6)
D_2	16.923603267895	17.195543463056	17.208492441230	17.218929768087	17.227194606456
$\lvert\frac{\Delta D_2}{D_2}\rvert$	0.182(-1)	0.246(-2)	0.171(-2)	0.110(-2)	0.622(-3)
D_3	-10.31963392978	-8.066136032966	-8.116802472857	-8.072012037121	-8.073689995482
$\lvert\frac{\Delta D_3}{D_3}\rvert$	0.279	0.629(-3)	0.565(-2)	0.987(-4)	0.307(-3)
β_n	0.157(-4)	0.737(-7)	-0.976(-8)	-0.467(-8)	-0.464(-9)
$\|\mathbf{x}\|$	435	0.129(5)	0.790(6)	0.166(9)	0.445(13)
$\|\mathbf{b}\|$	112	79.1	70.7	64.6	55.9
σ_{max}	0.997	0.979	0.975	0.973	0.970
σ_{min}	0.611(-6)	0.585(-11)	0.131(-13)	0.373(-16)	0.238(-21)
Cond	0.163(7)	0.167(12)	0.746(14)	0.261(17)	0.407(22)
Cond_eff	0.421(6)	0.104(10)	0.684(10)	0.104(11)	0.528(11)
Cond_EE	0.711(7)	0.107(10)	0.727(10)	0.138(11)	0.121(12)

Table 1: The errors and condition numbers by the MFS for $R = \sqrt{3}$ and $L = 3$, where M is the number of Gaussian nodes on $\overline{AB}$.

L	1	2	3	4
$\Vert\varepsilon\Vert_B$	0.431(-4)	0.119(-5)	0.198(-7)	0.407(-9)
D_0	401.13173081557	401.16280525410	401.16243448162	401.16245348347
$\vert\frac{\Delta D_0}{D_0}\vert$	0.766(-4)	0.876(-6)	0.480(-7)	0.653(-9)
D_1	87.626121504438	87.713968940637	87.655463873408	87.655882773581
$\vert\frac{\Delta D_1}{D_1}\vert$	0.340(-3)	0.662(-3)	0.521(-5)	0.427(-6)
D_2	–	17.171202178929	17.208492441230	17.237739905330
$\vert\frac{\Delta D_2}{D_2}\vert$	–	0.387(-2)	0.171(-2)	0.102(-4)
D_3	–	–	-8.116802472857	-8.084520000876
$\vert\frac{\Delta D_3}{D_3}\vert$	–	–	0.565(-2)	0.165(-2)
D_4	–	–	–	1.4267531744418
$\vert\frac{\Delta D_4}{D_4}\vert$	–	–	–	0.939(-2)
β_n	0.327(-5)	-0.370(-6)	-0.976(-8)	-0.563(-9)
$\Vert\mathbf{x}\Vert$	0.662(9)	0.626(8)	0.790(6)	0.495(5)
$\Vert\mathbf{b}\Vert$	70.7	70.7	70.7	70.7
σ_{max}	97.3	97.4	97.5	98.4
σ_{min}	0.161(-13)	0.161(-13)	0.131(-13)	0.116(-13)
Cond	0.604(14)	0.606(14)	0.746(14)	0.852(14)
Cond_eff	0.663(7)	0.703(8)	0.684(10)	0.124(12)
Cond_EE	0.217(8)	0.191(9)	0.727(10)	0.126(12)

Table 2: The errors and condition numbers by the MFS for $R = \sqrt{3}$, $M = 50$ and $N = 70$.

N	10	30	50	70	90
$\|\varepsilon\|_B$	0.778(-1)	0.141(-4)	0.113(-6)	0.198(-7)	0.524(-8)
D_0	402.46309997517	401.16134851351	401.16238204844	401.16243448162	401.16244622483
$\|\frac{\Delta D_0}{D_0}\|$	0.324(-2)	0.276(-5)	0.179(-6)	0.480(-7)	0.188(-7)
D_1	79.319577512882	87.653555483075	87.654492126134	87.655463873408	87.655712554666
$\|\frac{\Delta D_1}{D_1}\|$	0.951(-1)	0.269(-4)	0.163(-4)	0.521(-5)	0.237(-5)
D_2	17.038265058526	16.966100386233	17.180841526456	17.208492441230	17.219796251098
$\|\frac{\Delta D_2}{D_2}\|$	0.116(-1)	0.158(-1)	0.331(-2)	0.171(-2)	0.105(-2))
D_3	-3.269798138405	-7.835363761145	-8.160804814899	-8.116802472857	-8.100411215190
$\|\frac{\Delta D_3}{D_3}\|$	0.594	0.292(-1)	0.111(-1)	0.565(-2)	0.362(-2)
β_n	0.169(-1)	0.438(-5)	-0.774(-7)	-0.976(-8)	-0.195(-8)
$\|\mathbf{x}\|$	4226	437	1576	0.790(6)	0.722(9)
$\|\mathbf{b}\|$	70.7	70.7	70.7	70.7	70.7
σ_{max}	0.403	0.651	0.829	0.975	1.102
σ_{min}	0.509(-3)	0.177(-6)	0.544(-10)	0.131(-13)	0.294(-17)
Cond	791	0.368(7)	0.152(11)	0.746(14)	0.375(18)
Cond_eff	0.328(3)	0.914(6)	0.824(9)	0.684(10)	0.333(11)
Cond_EE	0.419(4)	0.161(8)	0.914(9)	0.727(10)	0.363(11)

Table 3: The errors and the condition numbers by the MFS for $R = \sqrt{3}$, $L = 3$ and $M = 50$.

R	1.12	1.32	1.52	$\sqrt{3}$	$\sqrt{4}$	$\sqrt{5}$
$\|\varepsilon\|_B$	0.115(-2)	0.161(-6)	0.162(-7)	0.198(-7)	0.228(-7)	0.244(-7)
D_0	401.15847235889	401.16245575474	401.16243699429	401.16243448162	401.16243256159	401.16243157319
$\|\frac{\Delta D_0}{D_0}\|$	0.993(-5)	0.501(-8)	0.418(-7)	0.480(-7)	0.528(-7)	0.553(-7)
D_1	84.268692851218	87.657744623531	87.655478652894	87.655463873408	87.655459414091	87.655460785067
$\|\frac{\Delta D_1}{D_1}\|$	0.386(-1)	0.208(-4)	0.504(-5)	0.521(-5)	0.526(-5)	0.524(-5)
D_2	19.564276176289	17.214577317415	17.210742606202	17.208492441230	17.206838709689	17.205999072258
$\|\frac{\Delta D_2}{D_2}\|$	0.135	0.135(-2)	0.158(-2)	0.171(-2)	0.180(-2)	0.185(-2)
D_3	-4.620916169199	-7.769378210887	-8.121075039850	-8.116802472857	-8.112957327852	-8.110543508143
$\|\frac{\Delta D_3}{D_3}\|$	0.428	0.374(-1)	0.618(-2)	0.565(-2)	0.517(-2)	0.487(-2)
β_n	0.110(-3)	0.506(-8)	-0.899(-8)	-0.976(-8)	-0.949(-8)	0.902(-8)
$\|\mathbf{x}\|$	0.103(7)	0.376(4)	0.351(5)	0.790(6)	0.328(8)	0.719(9)
$\|\mathbf{b}\|$	70.7	70.7	70.7	70.7	70.7	70.7
σ_{max}	0.948	0.812	0.899	0.975	1.193	1.370
σ_{min}	0.135(-9)	0.482(-11)	0.268(-12)	0.131(-13)	0.324(-15)	0.145(-16)
Cond	0.704(10)	0.168(12)	0.304(13)	0.746(14)	0.369(16)	0.947(17)
Cond_eff	0.509(6)	0.389(10)	0.751(10)	0.684(10)	0.666(10)	0.680(10)
Cond_EE	0.641(6)	0.139(11)	0.786(10)	0.727(10)	0.745(10)	0.784(10)

Table 4: The errors and the condition numbers by the MFS for $L = 3$, $M = 50$ and $N = 70$.

D_0	401.16243448161	D_1	87.65546387340
D_2	17.208492441230	D_3	-8.11680247285
c_1	3.49907401895	c_{36}	4.85132592170
c_2	2.10738562015	c_{37}	-15.9858068773
c_3	5.64350088839	c_{38}	19.0449195021
c_4	2.49132221822	c_{39}	-28.2660711338
c_5	6.95624424425	c_{40}	65.0030963508
c_6	-1.05459991519	c_{41}	-142.774821268
c_7	11.73258715777	c_{42}	386.504892267
c_8	-13.2325644703	c_{43}	-1103.84201625
c_9	25.67810162957	c_{44}	3207.90456503
c_{10}	-17.7671542148	c_{45}	-8976.78563697
c_{11}	-59.5973878280	c_{46}	23225.8659786
c_{12}	337.8518486653	c_{47}	-53951.4359850
c_{13}	-1082.19236253	c_{48}	110055.459240
c_{14}	2537.288637368	c_{49}	-193877.091894
c_{15}	-4916.93933515	c_{50}	290977.990042
c_{16}	7909.332284076	c_{51}	-367863.028942
c_{17}	-11056.4723078	c_{52}	388017.020632
c_{18}	13081.19591725	c_{53}	-338817.589337
c_{19}	-12844.1567566	c_{54}	243351.844016
c_{20}	10895.45959844	c_{55}	-142776.605534
c_{21}	-8089.79537742	c_{56}	67634.7433767
c_{22}	5244.367569821	c_{57}	-25192.2863611
c_{23}	-3019.00756106	c_{58}	6829.26934522
c_{24}	1542.377454292	c_{59}	-920.003962037
c_{25}	-722.714835078	c_{60}	-293.835832664
c_{26}	317.5646834221	c_{61}	293.996294369
c_{27}	-131.711138310	c_{62}	-147.529229042
c_{28}	62.88293833542	c_{63}	65.1544254251
c_{29}	-24.8954913995	c_{64}	-25.8291148898
c_{30}	18.47406317941	c_{65}	10.8612026318
c_{31}	-8.70513568494	c_{66}	-9.33948409507
c_{32}	8.401430954445	c_{67}	-2.16483054255
c_{33}	-8.65965347548	c_{68}	-7.44116733838
c_{34}	4.415985145558	c_{69}	-0.57848218969
c_{35}	-12.6219492434	c_{70}	-1.22948822199

Table 5: All coefficients of the MFS for $R = \sqrt{3}$, $L = 3$, $M = 50$ and $N = 70$, where the order of c_i corresponds to that of Q_i shown in Figure **??**.

$k(\sigma_k)$	74	73	61	55	54	53
$\|\varepsilon\|_B$	0.198(-7)	0.221(-7)	0.981(-7)	0.184(-6)	0.185(-6)	0.229(-6)
$\|\mathbf{x}_0 - \mathbf{x}_k\|$	0	0.744(6)	0.790(6)	0.790(6)	0.790(6)	0.790(6)
$\frac{\|\mathbf{x}_0 - \mathbf{x}_k\|}{\|\mathbf{x}_0\|}$	0	0.942	0.999	0.999	0.999	0.999
$\|\mathbf{x}_k\|$	0.790(6)	0.266(6)	0.691(3)	0.453(3)	0.452(3)	0.441(3)
σ_{max}	0.975	0.975	0.975	0.975	0.975	0.975
σ_k	0.131(-13)	0.165(-13)	0.942(-10)	0.126(-8)	0.134(-8)	0.245(-8)
Cond_k	0.746(14)	0.589(14)	0.104(11)	0.773(9)	0.727(9)	0.398(9)
Cond_eff_k	0.684(10)	0.161(11)	0.109(10)	0.124(9)	0.117(9)	0.655(9)

Table 6: Using TSVD by the MFS adding singular function for Motz's problem with $M = 50, N = 70$ and σ_k.

$m(\lambda = \sigma_m)$	74	73	61	55	54	53
$\|\varepsilon\|_B$	0.204(-7)	0.207(-7)	0.919(-7)	0.185(-6)	0.189(-6)	0.239(-6)
$\|\mathbf{x}_0 - \mathbf{x}_\lambda\|$	0.376(6)	0.463(6)	0.790(6)	0.790(6)	0.790(6)	0.790(6)
$\frac{\|\mathbf{x}_0 - \mathbf{x}_\lambda\|}{\|\mathbf{x}_0\|}$	0.476	0.586	0.999	0.999	0.999	0.999
$\|\mathbf{x}_\lambda\|$	0.443(6)	0.369(6)	0.689(3)	0.443(3)	0.443(3)	0.435(3)
σ_{max}	0.975	0.975	0.975	0.975	0.975	0.975
σ_m	0.131(-13)	0.165(-13)	0.942(-10)	0.126(-8)	0.134(-8)	0.245(-8)
Cond_λ	0.373(14)	0.295(14)	0.518(10)	0.387(10)	0.363(9)	0.199(9)
Cond_eff_λ	0.611(10)	0.579(10)	0.544(9)	0.633(8)	0.596(8)	0.332(8)

Table 7: Using Tikhonov by the MFS adding singular function for Motz's problem with $M = 50, N = 70$ and $\lambda = \sigma_m$.

N	28	56	70	84	112
M	20	40	50	60	80
$\|\varepsilon\|_B$	0.158(-2)	0.981(-6)	0.793(-7)	0.201(-7)	0.453(-8)
D_0	401.12887573859	401.16240970608	401.16239398651	401.16243770645	401.16244789522
$\frac{\Delta D_0}{D_0}$	0.837(-4)	0.109(-6)	0.149(-6)	0.399(-7)	0.146(-7)
D_1	88.221518029163	87.657999949896	87.654450862899	87.656100532179	87.655942266710
$\frac{\Delta D_1}{D_1}$	0.645(-2)	0.237(-4)	0.167(-4)	0.206(-5)	0.252(-6)
D_2	13.527929725644	17.181685547191	17.186583618000	17.207530863368	17.219967966057
$\frac{\Delta D_2}{D_2}$	0.215	0.326(-2)	0.298(-2)	0.176(-2)	0.104(-2)
D_3	-7.542280505172	-7.878679975644	-8.199899252808	-8.011174636452	-8.040059603255
$\frac{\Delta D_3}{D_3}$	0.655(-1)	0.239(-1)	0.159(-1)	0.744(-2)	0.386(-2)
β_n	0.159(-4)	-0.234(-7)	-0.768(-8)	0.687(-9)	-0.427(-9)
$\|\mathbf{x}\|$	0.113(4)	0.312(6)	0.250(8)	0.675(10)	0.116(16)
$\|\mathbf{b}\|$	111.8	79.06	70.71	64.55	55.90
σ_{max}	0.824	0.792	0.785	0.781	0.776
σ_{min}	0.221(-6)	0.112(-12)	0.523(-15)	0.291(-18)	0.457(-24)
Cond	0.373(7)	0.709(13)	0.150(16)	0.268(19)	0.169(25)
Cond_eff	0.449(6)	0.227(10)	0.540(10)	0.328(11)	0.105(12)

Table 8: The errors and condition numbers by the MFS for the source points uniformly on ℓ_{ell} with $a = \sqrt{\frac{12}{3}}$.

N	24	60	72	84	108
M	20	40	50	60	80
$\|\varepsilon\|_B$	0.224(-2)	0.831(-7)	0.144(-7)	0.621(-8)	0.158(-8)
D_0	401.18859196179	401.16242706571	401.16244175893	401.16244697676	401.16245114868
$\frac{\Delta D_0}{D_0}$	0.652(-4)	0.665(-7)	0.299(-7)	0.169(-7)	0.647(-8)
D_1	87.179421757914	87.655872693025	87.655893557855	87.655884210314	87.655887484528
$\frac{\Delta D_1}{D_1}$	0.544(-2)	0.542(-6)	0.304(-6)	0.411(-6)	0.373(-6)
D_2	22.507885620329	17.201051481221	17.213193974664	17.219408049237	17.226552028569
$\frac{\Delta D_2}{D_2}$	0.306	0.214(-2)	0.143(-2)	0.107(-2)	0.659(-3)
D_3	-4.488908470017	-8.038758866759	-8.044084655766	-8.052257985616	-8.062250475450
$\frac{\Delta D_3}{D_3}$	0.444	0.402(-2)	0.336(-2)	0.235(-2)	0.111(-2)
β_n	0.499(-3)	0.317(-7)	-0.158(-8)	0.875(-10)	-0.961(-11)
$\|\mathbf{x}\|$	413.5	497.9	1081	8938	0.145(7)
$\|\mathbf{b}\|$	111.8	79.06	70.71	64.55	55.90
σ_{max}	0.737	0.771	0.749	0.735	0.717
σ_{min}	0.389(-4)	0.183(-9)	0.213(-11)	0.308(-13)	0.315(-16)
Cond	0.189(5)	0.422(10)	0.351(12)	0.239(14)	0.227(17)
Cond_eff	0.695(4)	0.869(8)	0.306(11)	0.235(12)	0.123(13)

Table 9: The errors and condition numbers by the MFS for the source points uniformly on ℓ_{rect} with $b = 0.5$.

M	20	40	80	160
$\|\varepsilon\|_B$	1.778	1.268	0.691	0.364
D_0	363.62110612216	374.03325412650	379.06683897012	384.13115632261
$\|\frac{\Delta D_0}{D_0}\|$	0.936(-1)	0.676(-1)	0.551(-1)	0.425(-1)
D_1	124.45339770325	111.54973130503	104.27171752277	99.887546907515
$\|\frac{\Delta D_1}{D_1}\|$	0.419	0.273	0.189	0.139
D_2	17.740812763461	18.922458758838	19.810368741251	19.467611719889
$\|\frac{\Delta D_2}{D_2}\|$	0.292(-1)	0.977(-1)	0.149	0.129
D_3	-5.598209645819	-5.690456850037	-5.524165452999	-5.998038734794
$\|\frac{\Delta D_3}{D_3}\|$	0.306	0.295	0.316	0.257
β_n	-0.151	-0.217	0.535(-1)	-0.446(-1)
$\|\mathbf{x}\|$	0.388(6)	0.197(6)	0.618(5)	0.613(5)
$\|\mathbf{b}\|$	111.8	111.8	111.8	111.8
σ_{max}	0.962	0.889	0.871	0.867
σ_{min}	0.149(-5)	0.111(-5)	0.896(-6)	0.805(-6)
Cond	0.648(6)	0.799(6)	0.972(6)	0.108(7)
Cond_eff	194	510	0.202(4)	0.227(4)
Cond_EE	742	516	0.209(4)	0.251(4)

Table 10: The error norms and condition numbers by the MFS for Motz's problem with $R = \sqrt{3}, N = 28$ and $\alpha = 1$.

q	1	2	3	4	5
$\|\varepsilon\|_B$	0.364	0.437(-1)	0.115(-1)	0.877(-2)	0.875(-2)
D_0	384.13115632261	387.70589280210	401.41867393852	402.09184254292	402.095311660866
$\|\frac{\Delta D_0}{D_0}\|$	0.425(-1)	0.336(-1)	0.639(-3)	0.232(-2)	0.233(-2)
D_1	99.887546907515	97.271108671312	86.037835695430	85.486201212337	85.4833584055417
$\|\frac{\Delta D_1}{D_1}\|$	0.139	0.109	0.185(-1)	0.248(-1)	0.248(-1)
D_2	19.467611719889	18.929865670286	17.285255871533	17.204611447517	17.2041958555404
$\|\frac{\Delta D_2}{D_2}\|$	0.129	0.982(-1)	0.275(-2)	0.193(-2)	0.196(-2)
D_3	-5.998038734794	-6.383564207251	-7.648528464882	-7.710600862676	-7.7109207469204
$\|\frac{\Delta D_3}{D_3}\|$	0.257	0.209	0.524(-1)	0.447(-1)	0.446(-1)
β_n	-0.446(-1)	-0.877(-1)	-0.157(-1)	-0.141(-1)	-0.141(-1)
$\|\mathbf{x}\|$	0.613(5)	0.146(6)	0.546(5)	0.502(5)	0.502(5)
$\|\mathbf{b}\|$	111.8	111.8	111.8	111.8	111.8
σ_{max}	0.867	0.865	0.865	0.865	0.865
σ_{min}	0.805(-6)	0.605(-6)	0.299(-6)	0.293(-6)	0.293(-6)
Cond	0.108(7)	0.143(7)	0.289(7)	0.295(7)	0.295(7)
Cond_eff	0.227(4)	0.127(4)	0.685(4)	0.758(4)	0.759(4)
Cond_EE	0.251(4)	0.128(4)	0.712(4)	0.794(4)	0.795(4)

Table 11: The error norms and condition numbers from the pure MFS for Motz's problem for $R = \sqrt{3}$, $M = 20$ and $N = 28$.

N	28	42	56	70	84
M	20	30	40	50	60
$\|\varepsilon\|_B$	0.877(-2)	0.214(-3)	0.864(-5)	0.274(-6)	0.130(-7)
D_0	402.09184254292	405.63488484792	401.64565381496	403.901058544929	401.504514344611
$\|\frac{\Delta D_0}{D_0}\|$	0.232(-2)	0.112(-1)	0.120(-2)	0.683(-2)	0.853(-3)
D_1	85.486201212337	83.156628936543	86.5255205015507	84.918939324102	86.8840334693767
$\|\frac{\Delta D_1}{D_1}\|$	0.248(-1)	0.513(-1)	0.129(-1)	0.312(-1)	0.881(-2)
D_2	17.204611447517	16.345251566472	16.7878829220685	16.318930673832	16.6395941822623
$\|\frac{\Delta D_2}{D_2}\|$	0.193(-2)	0.518(-1)	0.261(-1)	0.533(-1)	0.347(-1)
D_3	-7.710600862676	-8.140490333528	-7.8077693048529	-8.052717961510	-7.8482111479329
$\|\frac{\Delta D_3}{D_3}\|$	0.447(-1)	0.858(-2)	0.326(-1)	0.229(-2)	0.276(-1)
β_n	-0.141(-1)	-0.535(-3)	0.139(-4)	0.698(-6)	-0.213(-7)
$\|\mathbf{x}\|$	0.502(5)	0.214(7)	0.935(8)	0.471(10)	0.220(12)
$\|\mathbf{b}\|$	112	91.3	79.1	70.7	64.5
σ_{max}	0.865	0.865	0.865	0.865	0.865
σ_{min}	0.293(-6)	0.257(-9)	0.153(-11)	0.150(-15)	0.983(-19)
Cond	0.295(7)	0.337(10)	0.566(13)	0.576(16)	0.880(19)
Cond_eff	0.758(4)	0.166(6)	0.553(7)	0.999(8)	0.298(10)
Cond_EE	0.794(4)	0.171(6)	0.566(7)	0.101(9)	0.303(10)

Table 12: The error norms and condition numbers from the pure MFS for Motz's problem for $R = \sqrt{3}$, $\alpha = 4$.

D_0	401.504514344611	D_1	86.8840334693767
D_2	16.6395941822623	D_3	-7.8482111479329
c_1	.491387321370D+03	c_{43}	-.51752378965D+02
c_2	-.48322716360D+03	c_{44}	.632198747372D+02
c_3	.498631040500D+03	c_{45}	-.80643532201D+02
c_4	-.62046428611D+03	c_{46}	.114822228337D+03
c_5	.852602439285D+03	c_{47}	-.87211046751D+02
c_6	-.14215918724D+04	c_{48}	-.19754028332D+03
c_7	.269410878337D+04	c_{49}	.252407890798D+04
c_8	-.58717119691D+04	c_{50}	-.17846361028D+05
c_9	.138814930701D+05	c_{51}	.117362488215D+06
c_{10}	-.34384360457D+05	c_{52}	-.74528202326D+06
c_{11}	.853116587961D+05	c_{53}	.449610577197D+07
c_{12}	-.20521086079D+06	c_{54}	-.25010806755D+08
c_{13}	.465947174702D+06	c_{55}	.124589584910D+09
c_{14}	-.97991449292D+06	c_{56}	-.54202238950D+09
c_{15}	.188290692614D+07	c_{57}	.201835718510D+10
c_{16}	-.32746205325D+07	c_{58}	-.63340241810D+10
c_{17}	.512088955005D+07	c_{59}	.165586759379D+11
c_{18}	-.71686899498D+07	c_{60}	-.35761144558D+11
c_{19}	.895598609608D+07	c_{61}	.634402739441D+11
c_{20}	-.99648035454D+07	c_{62}	-.92121325213D+11
c_{21}	.986061783032D+07	c_{63}	.109312719485D+12
c_{22}	-.86704303039D+07	c_{64}	-.10599564313D+12
c_{23}	.677226463301D+07	c_{65}	.841202271679D+11
c_{24}	-.46990352789D+07	c_{66}	-.54819461868D+11
c_{25}	.289846804493D+07	c_{67}	.294891303375D+11
c_{26}	-.15916310560D+07	c_{68}	-.13193813281D+11
c_{27}	.780158417112D+06	c_{69}	.496146614829D+10
c_{28}	-.34275972008D+06	c_{70}	-.15904226582D+10
c_{29}	.135888200678D+06	c_{71}	.442780363168D+09
c_{30}	-.49061962295D+05	c_{72}	-.10969852687D+09
c_{31}	.163712986752D+05	c_{73}	.249510976517D+08
c_{32}	-.51215232444D+04	c_{74}	-.54180381325D+07
c_{33}	.154511906094D+04	c_{75}	.117767331983D+07
c_{34}	-.44507188147D+03	c_{76}	-.270444688924D+06
c_{35}	.125418459170D+03	c_{77}	.6932717850916D+05
c_{36}	-.25461515062D+02	c_{78}	-.208194120307D+05
c_{37}	-.61884052928D+01	c_{79}	.7519822051232D+04
c_{38}	.145763630024D+02	c_{80}	-.332107479234D+04
c_{39}	-.25060228690D+02	c_{81}	.1723748649982D+04
c_{40}	.239726174582D+02	c_{82}	-.107905955330D+04
c_{41}	-.35894541412D+02	c_{83}	.7343071927688D+03
c_{42}	.366354824136D+02	c_{84}	-.589529464196D+03

Table 13: All coefficients by the MFS for Motz's problem with $R = \sqrt{3}$, $\sigma = 3$, $M = 60$ and $N = 84$, where the order of c_i is shown Figure 5.

$k(\sigma_k)$	70	69	68	67	66	65
$\|\varepsilon\|_B$	0.274(-6)	0.749(-6)	0.139(-5)	0.306(-5)	0.332(-5)	0.614(-5)
$\|\mathbf{x}_0 - \mathbf{x}_k\|$	0	0.464(10)	0.471(10)	0.471(10)	0.471(10)	0.471(10)
$\frac{\|\mathbf{x}_0 - \mathbf{x}_k\|}{\|\mathbf{x}_0\|}$	0	0.986	0.999	0.999	0.999	0.999
$\|\mathbf{x}_k\|$	0.471(10)	0.796(9)	0.218(9)	0.102(9)	0.667(8)	0.259(8)
σ_{max}	0.865	0.865	0.865	0.865	0.865	0.865
σ_k	0.150(-15)	0.153(-14)	0.141(-13)	0.167(-13)	0.842(-13)	0.123(-12)
Cond_k	0.576(16)	0.567(15)	0.613(14)	0.518(14)	0.103(14)	0.706(13)
Cond_eff_k	0.999(8)	0.582(8)	0.229(8)	0.415(8)	0.126(8)	0.223(8)

Table 14: Using TSVD by the MFS for Motz's problem with $M = 50, N = 70$ and σ_k.

$m(\lambda = \sigma_m)$	70	69	68	67	66	65
$\|\varepsilon\|_B$	0.444(-6)	0.946(-6)	0.201(-5)	0.221(-5)	0.416(-5)	0.487(-5)
$\|\mathbf{x}_0 - \mathbf{x}_\lambda\|$	0.232(10)	0.461(10)	0.471(10)	0.471(10)	0.471(10)	0.471(10)
$\frac{\|\mathbf{x}_0 - \mathbf{x}_\lambda\|}{\|\mathbf{x}_0\|}$	0.493	0.980	0.999	0.999	0.999	0.999
$\|\mathbf{x}_\lambda\|$	0.245(10)	0.442(9)	0.125(9)	0.110(9)	0.399(8)	0.311(8)
σ_{max}	0.865	0.865	0.865	0.865	0.865	0.865
σ_m	0.150(-15)	0.153(-14)	0.141(-13)	0.167(-13)	0.842(-13)	0.123(-12)
Cond_λ	0.288(16)	0.283(15)	0.307(14)	0.259(14)	0.514(13)	0.353(13)
Cond_eff_λ	0.959(8)	0.525(8)	0.200(8)	0.192(8)	0.105(8)	0.927(7)

Table 15: Using Tikhonov regularization by the MFS for Motz's problem with $M = 50, N = 70$ and $\lambda = \sigma_m$.

Part II

Truncated Singular Value Decomposition and Tikhonov Regularization

8 Introduction about TSVD and TR

Below, we begin with TSVD and TR for MFS. Consider the over-determined system of linear algebraic equations

$$\mathbf{F}\,\mathbf{x} = \mathbf{b}, \tag{8.1}$$

where $\mathbf{F} \in R^{m\times n}$ $(m \geq n)$, $\mathbf{x} \in R^n$ and $\mathbf{b} \in R^m$. When there exists the perturbation of $\mathbf{F}$ and $\mathbf{b}$, there are the equalities,

$$\mathbf{F}(\mathbf{x} + \boldsymbol{\Delta}\mathbf{x}) = \mathbf{b} + \boldsymbol{\Delta}\mathbf{b}, \tag{8.2}$$

$$(\mathbf{F} + \boldsymbol{\Delta}\mathbf{F})(\mathbf{x} + \boldsymbol{\Delta}\mathbf{x}) = \mathbf{b} + \boldsymbol{\Delta}\mathbf{b}, \tag{8.3}$$

where $\boldsymbol{\Delta}\mathbf{F} \in R^{m\times n}$, $\boldsymbol{\Delta}\mathbf{x} \in R^n$ and $\boldsymbol{\Delta}\mathbf{b} \in R^m$.

To measure the perturbation solutions from rounding errors, the traditional condition number in 2-norm is a useful tool, defined by

$$\text{Cond} = \frac{\sigma_{max}}{\sigma_{min}}, \tag{8.4}$$

where σ_{max} and σ_{min} are the maximal and the minimal singular values of matrix $\mathbf{F}$, respectively. For (8.2), there exists the bound,

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \text{Cond}\frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}, \tag{8.5}$$

where $\|\cdot\|$ is the 2-norm. For (8.3), Cond in (8.4) may also be used to provide the perturbation errors $\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|}$ with a constant factor.

Recently, the effective condition number is proposed in [27], defined by

$$\text{Cond\_eff} = \frac{\|\mathbf{b}\|}{\sigma_{min}\|\mathbf{x}\|}, \tag{8.6}$$

to also give the bound

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \text{Cond\_eff}\frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}, \tag{8.7}$$

The Cond_eff is smaller, and even much smaller than the traditional Cond. Compared Cond_eff with Cond, the σ_{min} is intrinsic to stability, but not σ_{max}. In practice, some numerical methods, such as the spectral methods, the fundamental solution methods (FSM), the radial basis function method, etc. are often very ill-conditioning, i.e., the σ_{min} is infinitesimal. In this case, both Cond and Cond_eff are large. To reduce the severe instability, two techniques can be solicited: (1) the truncated singular value decomposition, and (2) the Tikhonov regularization. Both the TSVD and Tikhonov regularization play a rule of filtering, and are successful for noise reduction in the least squares (LS) problems. Let us mention the related important references. The truncated SVD are discussed in Hansen [24], and Chan and Hansen [7], and introduced in Chen et al. [8, 9]. The Tikhonov regularization was first proposed by Tikhonov in [48] in 1963, introduced in Tikhonov and Arsenin [49], and analyzed in Hansen [23]. Recently, the truncated SVD and Tikhonov regularization are studied together in [7, 24], and Fierro et al. [21] for noise reduction.

This thesis is organized as follows. In Section 9, the algorithms of the TSVD and the TR are described. In Section 10, the analysis of condition number and effective condition number is made, and Section 11, error bounds for TSVD and TR are derived. In Section 12, the combination of TSVD and TR is explored, and in the last section, numerical experiments are reported.

9 Algorithms

In this thesis, we assume that Rank($\mathbf{F}$) $= n$. The singular value decomposition of $\mathbf{F}$ is given by

$$\mathbf{A} = \mathbf{U}\,\boldsymbol{\Sigma}\,\mathbf{V}^T\,, \tag{9.1}$$

where $\mathbf{U} \in R^{m\times m}$ and $\mathbf{V} \in R^{n\times n}$, and $\boldsymbol{\Sigma} \in R^{m\times n}$ is diagonal with a positive singular values

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0\,, \tag{9.2}$$

where we also denote $\sigma_1 = \sigma_{max}$ and $\sigma_n = \sigma_{min}$. In this thesis, we also assume that $\sigma_{min} \ll \sigma_{max}$. Denote $\mathbf{U} = (\mathbf{u}_1, \cdots, \mathbf{u}_m)$ and $\mathbf{V} = (\mathbf{v}_1, \cdots, \mathbf{v}_n)$, where $\mathbf{u}_i \in R^m$ and $\mathbf{v}_i \in R^n$ are vectors. Then the least squares method (LSM) for (8.1) can be expressed by

$$\mathbf{x_0} = \mathbf{x_{LSM}} = \sum_{i=1}^{n} \frac{\beta_i}{\sigma_i}\mathbf{v}_i\,, \tag{9.3}$$

where $\beta_i = \mathbf{u}_i^T\mathbf{b}$. When σ_n is infinitesimal, the solution $\mathbf{x_0}$ in (9.3) may be large and even huge if $\beta_n \neq 0$. Also when $\mathbf{v}_n$ is highly oscillating, the solution $\mathbf{x_0}$ is also highly oscillating. One way is to discard the part of (9.3) involving very small σ_i, e.g.,

$$\sigma_i < \lambda\,, \tag{9.4}$$

we obtain the truncated least squares solution (TLSS)

$$\mathbf{x}_k = \sum_{i=1}^{k} \frac{\beta_i}{\sigma_i}\mathbf{v}_i, \qquad \sigma_k \geq \lambda \geq \sigma_{k+1}\,. \tag{9.5}$$

The other approach to deal with the very small σ_n is the Tikhonov regularization. Consider the following minimization problem with a parameter λ,

$$\min_{\mathbf{x}\in R^n} \left\{ \|\mathbf{A}\,\mathbf{x} - \mathbf{b}\|^2 + \lambda^2\|\mathbf{L}\mathbf{x}\|^2 \right\}\,, \tag{9.6}$$

where the matrix $\mathbf{L} \in R^{p\times n}$, $p \leq n$. When $\mathbf{L}$ is the identifying matrix $\mathbf{I} \in R^{n\times n}$, Eq. (9.6) leads to

$$\min_{\mathbf{x}\in R^n} \left\{ \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 + \lambda^2\|\mathbf{x}\|^2 \right\}, \tag{9.7}$$

which is the standard form of Tikhonov regularization. In fact, we have from (9.3)

$$\|\mathbf{x_0}\|^2 = \sum_{i=1}^{n} \left(\frac{\beta_i}{\sigma_i}\right)^2 ,$$

when $\beta_n \neq 0$ and $\sigma_n = \sigma_{min} \longrightarrow 0$, we have $\|\mathbf{x_0}\| \longrightarrow \infty$. The equation (9.7) is to confine $\|\mathbf{x}\|$ not to be large, so that to guarantee a better stability, against the case $\|\mathbf{x}\| \longrightarrow \infty$. Since the solution of (9.7) can be expressed by

$$\mathbf{x}_\lambda = \left(\mathbf{A}^T\mathbf{A} + \lambda^2\mathbf{I}\right)^{-1} \mathbf{A}^T\mathbf{b} . \tag{9.8}$$

Hence, a batter stability against the huge $\|\mathbf{x}\|$ from $\sigma_{min} \longrightarrow 0$ can be succeeded. By the SVD in (9.1) can have

$$\mathbf{x}_\lambda = \sum_{i=1}^{n} \frac{\sigma_i}{\sigma_i^2 + \lambda^2} \beta_i \, \mathbf{v}_i . \tag{9.9}$$

In this thesis, we assume that the parameter λ satisfies [4]

$$\sigma_{min} \leq \lambda \leq \sigma_{max}, \qquad \sigma_{min} \ll \sigma_{max} . \tag{9.10}$$

The equation (9.9) is called the Tikhonov regularization. Both (9.5) and (9.9) may overcome the drawback of the infinite simile σ_{min}. When $\mathbf{v}_{min} = \mathbf{v}_n$ is of high frequency, the solution (9.5) by TSVD may remove the projection of $\mathbf{x}_0$ in $\mathbf{v}_{min}$, but the solution (9.9) is not. In order to remove the projection of $\mathbf{x}_\lambda$ in $\mathbf{v}_{min}$, we propose the new combination of truncated Tikhonov regularization, to seek the solution

$$\mathbf{x}_{k-\lambda} = \sum_{i=1}^{k} \frac{\sigma_i}{\sigma_i^2 + \lambda^2} \beta_i \mathbf{v}_i , \quad k \leq n , \tag{9.11}$$

where λ satisfies

$$\sigma_k \leq \lambda \leq \sigma_{max} . \tag{9.12}$$

Evidently, the solution (9.11) has the advantage of $\mathbf{x}_k$ to remove completely the high frequencies, when $k < n$.

[4] Ajustification for (9.10) is given in Remark 11.1 later.

10 Effective Condition Number for Least Squares Methods with Rank Deficient

For the TSVD of (9.5), the condition number and the effective condition number are given by

$$\text{Cond}_k = \frac{\sigma_1}{\sigma_k}, \tag{10.1}$$

and

$$\text{Cond\_eff}_k = \frac{\|\mathbf{b}\|}{\sigma_k \|\mathbf{x}_k\|}. \tag{10.2}$$

when $k = n$, $\text{Cond}_n = \text{Cond}$ and $\text{Cond\_eff}_n = \text{Cond\_eff}$. Evidently, Cond_eff_k in (10.2) is smaller or much smaller than Cond_k in (10.1).

Below, consider (9.9). Denote the singular values of the matrix of the Tikhonov regularization

$$\mu_i = \sigma_i + \frac{\lambda^2}{\sigma_i}. \tag{10.3}$$

Then Eq. (9.9) is rewritten as

$$\mathbf{x}_\lambda = \sum_{i=1}^{n} \frac{\beta_i}{\mu_i} \mathbf{v}_i. \tag{10.4}$$

The condition number and the effective condition number are given by

$$\text{Cond}_\lambda = \frac{\max_i \mu_i}{\min_i \mu_i}, \tag{10.5}$$

and

$$\text{Cond\_eff}_\lambda = \frac{\|\mathbf{b}\|}{(\min_i \mu_i)\|\mathbf{x}\|}. \tag{10.6}$$

Based on the denfinitions in (10.1), (10.2), (10.5) and (10.6), we obtain

$$\text{Cond\_eff}_k \leq \text{Cond}_k, \tag{10.7}$$

$$\text{Cond\_eff}_\lambda \leq \text{Cond}_\lambda. \tag{10.8}$$

Hence the effective condition number is also smaller than condition number.

First, we have the following lemma.

Lemma 10.1 *Let (9.10) hold for μ_i in (10.3)*

$$\mu_{min} \;=\; \min_i \mu_i \;=\; 2\lambda \tag{10.9}$$

Proof : Define a function

$$f(y) \;=\; y + \frac{\lambda^2}{y}\,. \tag{10.10}$$

The singular values μ_i in (10.3) are denoted by $\mu_i \;=\; f(\sigma_i)$. We may seek the extreme values of function $f(y)$ on $y \in [\sigma_{min}, \sigma_{max}]$, where $\sigma_1 \;=\; \sigma_{max}$ and $\sigma_n \;=\; \sigma_{min}$. First, the stationary point is given by

$$0 \;=\; f^{'}(\bar{y}) \;=\; 1 - \frac{\lambda^2}{\bar{y}^2}\,, \tag{10.11}$$

to give $\bar{y} \;=\; \lambda$. Since $f^{''}(\bar{y}) \;=\; \frac{2}{\lambda} > 0$, the minimum value of $f(y)$ is found by

$$\min f(y) = f(\bar{y}) \;=\; 2\lambda\,, \quad y \in [\sigma_{min}, \sigma_{max}]\,. \tag{10.12}$$

Hence we have

$$\min_i \mu_i \geq \min f(y) = 2\lambda\,, \quad y \in [\sigma_{min}, \sigma_{max}]\,. \tag{10.13}$$

This completes the proof of Lemma 10.1. ∎

Lemma 10.2 *Let (9.10) hold for μ_i in (10.3), there exist the bounds*

$$\max_i \mu_i \;=\; \sigma_{max} + \frac{\lambda^2}{\sigma_{max}}, \quad \text{if } \lambda \leq \sqrt{\sigma_{min}\sigma_{max}}\,, \tag{10.14}$$

and

$$\max_i \mu_i \;=\; \sigma_{min} + \frac{\lambda^2}{\sigma_{min}}, \quad \text{if } \lambda \geq \sqrt{\sigma_{min}\sigma_{max}}\,. \tag{10.15}$$

Proof : From Lemma 10.1, for $f(y)$ in (10.10) there exists one minimum value at $y = \lambda, \quad y \in [\sigma_{min}, \sigma_{max}]$, the maximum value may occur at the two boundary points, i.e.,

$$\max_{y \in [\sigma_{min}, \sigma_{max}]} f(y) = \max \{f(\sigma_{min}), f(\sigma_{max})\} , \tag{10.16}$$

where

$$f(\sigma_{max}) = \sigma_{max} + \frac{\lambda^2}{\sigma_{max}} , \quad f(\sigma_{min}) = \sigma_{min} + \frac{\lambda^2}{\sigma_{min}} . \tag{10.17}$$

To show (10.14), it is sufficient to prove

$$f(\sigma_{max}) \geq f(\sigma_{min}) , \tag{10.18}$$

i.e.,

$$\sigma_{max} + \frac{\lambda^2}{\sigma_{max}} \geq \sigma_{min} + \frac{\lambda^2}{\sigma_{min}} \tag{10.19}$$

which is equivalent to

$$\sigma_{max} - \sigma_{min} \geq \lambda^2 \left[\frac{1}{\sigma_{min}} - \frac{1}{\sigma_{max}} \right] = \lambda^2 \frac{\sigma_{max} - \sigma_{min}}{\sigma_{min}\sigma_{max}} , \tag{10.20}$$

which holds if and only if

$$\lambda \leq \sqrt{\sigma_{min}\sigma_{max}} . \tag{10.21}$$

This is the first desired result (10.14).

Similarly, to show (10.15) it is sufficient to prove

$$f(\sigma_{min}) \geq f(\sigma_{max}) , \tag{10.22}$$

i.e.,

$$\sigma_{min} + \frac{\lambda^2}{\sigma_{min}} \geq \sigma_{max} + \frac{\lambda^2}{\sigma_{max}} , \tag{10.23}$$

which is equivalent to

$$\sigma_{max} - \sigma_{min} \leq \frac{\lambda^2[\sigma_{max} - \sigma_{min}]}{\sigma_{min}\sigma_{max}}\,, \tag{10.24}$$

i.e.,

$$\lambda \geq \sqrt{\sigma_{min}\sigma_{max}}\,. \tag{10.25}$$

This is the second desired result (10.15), and this completes the proof of Lemma 10.2. ∎

From Lemmas 10.1 and 10.2 we have the following theorem.

Theorem 10.1 *Let (9.10) hold. The effective condition number and the condition number for Tikhonov regularization are given by*

$$\text{Cond\_eff}_\lambda = \frac{\|\mathbf{b}\|}{2\lambda\|\mathbf{x}_\lambda\|}\,, \tag{10.26}$$

$$\text{Cond}_\lambda = \frac{\sigma_{max}^2 + \lambda^2}{2\lambda\sigma_{max}} \quad \text{if} \quad \lambda \leq \sqrt{\sigma_{min}\sigma_{max}}\,, \tag{10.27}$$

$$\text{Cond}_\lambda = \frac{\sigma_{min}^2 + \lambda^2}{2\lambda\sigma_{min}} \quad \text{if} \quad \lambda \geq \sqrt{\sigma_{min}\sigma_{max}}\,. \tag{10.28}$$

Corollary 10.1 *There exist bounds,*

$$\frac{\|x_\lambda - \tilde{x}_\lambda\|}{\|x_\lambda\|} \leq \tilde{K}_\lambda \frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}_0\|}\,, \tag{10.29}$$

where $\Delta\mathbf{b}$ is a perturbation of vector $\mathbf{b}$, and $\mathbf{b}_0$ is the vector $\mathbf{b}$ in the spanned space of $\mathbf{A}$, and the condition number is given in [24] by

$$\tilde{K}_\lambda = \frac{\sigma_{max}}{\lambda} \quad \text{if} \quad \lambda \leq \sqrt{\sigma_{min}\sigma_{max}}\,, \tag{10.30}$$

$$\tilde{K}_\lambda = \frac{\lambda}{\sigma_{min}} \quad \text{if} \quad \lambda \geq \sqrt{\sigma_{min}\sigma_{max}}\,. \tag{10.31}$$

Proof : When $\lambda \leq \sqrt{\sigma_{min}\sigma_{max}}$, we have from Theorem 10.1 and (9.10)

$$\text{Cond}_\lambda = \frac{\sigma_{max}^2 + \lambda^2}{2\lambda\sigma_{max}} = \frac{2\sigma_{max}^2}{2\lambda\sigma_{max}} + \frac{\lambda^2 - \sigma_{max}^2}{2\lambda\sigma_{max}} \leq \frac{\sigma_{max}}{\lambda} = \tilde{K}_\lambda\,. \tag{10.32}$$

This is the first desired result (10.30)

Next, when $\lambda \geq \sqrt{\sigma_{min}\sigma_{max}}$, we have from Theorem 10.1 and (9.10)

$$\begin{aligned}\text{Cond}_\lambda &= \frac{\sigma_{min}}{2\lambda} + \frac{\lambda}{2\sigma_{min}} = \frac{\lambda}{\sigma_{min}} + \frac{1}{2}\left[\frac{\sigma_{min}}{\lambda} - \frac{\lambda}{\sigma_{min}}\right] \\ &= \frac{\lambda}{\sigma_{min}} + \frac{1}{2}\frac{\sigma_{min}^2 - \lambda^2}{\lambda\sigma_{min}} \leq \frac{\lambda}{\sigma_{min}} = \tilde{K}_\lambda\,. \end{aligned} \tag{10.33}$$

This is the second desired result (10.31) and completes, the proof of Corollary 10.1. ∎

Corollary 10.1 displays that the definitions of Cond_λ in (10.27) and (10.28) and $\hat{K}_\lambda$ in (10.30) and (10.31) in Hansen [24] are consistent to each other. However, from (10.32) and (10.33), Cond_λ in this thesis is sharper, to provide a lower bound: $\text{Cond}_\lambda < \tilde{K}_\lambda$, when $\sigma_{min} < \lambda < \sigma_{max}$.

Corollary 10.2 *When* $\lambda = \sigma_k \leq \sqrt{\sigma_{min}\sigma_{max}}$.

$$\text{Cond}_\lambda = \frac{\sigma_{max}}{2\sigma_k} + \frac{\sigma_k}{2\sigma_{max}} = \frac{\text{Cond}_k}{2} + \frac{1}{2\text{Cond}_k}\,. \tag{10.34}$$

and where $\lambda = \sigma_k \geq \sqrt{\sigma_{min}\sigma_{max}}$

$$\text{Cond}_\lambda = \frac{\sigma_{min}}{2\sigma_k} + \frac{\sigma_k}{2\sigma_{min}} = \frac{1}{2}\left(\frac{\text{Cond}_k}{\text{Cond}} + \frac{\text{Cond}}{\text{Cond}_k}\right)\,. \tag{10.35}$$

Proof : When $\lambda = \sigma_k \leq \sqrt{\sigma_{min}\sigma_{max}}$, we have from Theorem 10.1

$$\text{Cond}_\lambda = \frac{\sigma_{max}}{2\sigma_k} + \frac{\sigma_k}{2\sigma_{max}} = \frac{1}{2}\left(\text{Cond}_k + \frac{1}{\text{Cond}_k}\right)\,. \tag{10.36}$$

when $\lambda = \sigma_k \geq \sqrt{\sigma_{min}\sigma_{max}}$, we have from Theorem 10.1

$$\begin{aligned}\text{Cond}_\lambda &= \frac{\sigma_{min}}{2\sigma_k} + \frac{\sigma_k}{2\sigma_{min}} = \frac{1}{2}\left(\frac{\sigma_{min}}{\sigma_{max}}\cdot\frac{\sigma_{max}}{\sigma_k} + \frac{\sigma_{max}\sigma_k}{\sigma_{min}\sigma_{max}}\right) \\ &= \frac{1}{2}\left(\frac{\text{Cond}_k}{\text{Cond}} + \frac{\text{Cond}}{\text{Cond}_k}\right)\,. \end{aligned} \tag{10.37}$$

This completes the proof of Corollary 10.2. ∎

Corollary 6.4 analysis that when $\lambda = \sigma_k \leq \sqrt{\sigma_{min}\sigma_{max}}$, Cond_λ is about a half of Cond_k.

Corollary 10.3 *There exist the sounds*

$$\|\mathbf{x}_k\| \leq \frac{\|\mathbf{b}\|}{\lambda}, \tag{10.38}$$

$$\|\mathbf{x}_\lambda\| \leq \frac{\|\mathbf{b}\|}{2\lambda}. \tag{10.39}$$

Proof : Since $\sigma_i \geq \lambda$ for $i \geq k$, we have from (9.5)

$$\|\mathbf{x}_k\|^2 = \sum_{i=1}^{k} \frac{\beta_i^2}{\sigma_i^2} \leq \frac{1}{\lambda^2} \sum_{i=1}^{k} \beta_i^2 \leq \frac{1}{\lambda^2} \sum_{i=1}^{n} \beta_i^2 \leq \frac{1}{\lambda^2} \sum_{i=1}^{m} \beta_i^2 = \frac{1}{\lambda^2} \|\mathbf{b}\|^2, \tag{10.40}$$

where we have used

$$\|\mathbf{b}\| = \sqrt{\sum_{i=1}^{m} \beta_i^2}. \tag{10.41}$$

This is the first result (10.38).

Next, we have from Lemma 10.1 and (10.4),

$$\|\mathbf{x}_\lambda\|^2 = \sum_{i=1}^{k} \left(\frac{\beta_i}{\mu_i}\right)^2 \leq \frac{1}{(\min \mu_i)^2} \sum_{i=1}^{k} \beta_i^2 \leq \frac{1}{(2\lambda)^2} \sum_{i=1}^{n} \beta_i^2 \leq \frac{1}{(2\lambda)^2} \|\mathbf{b}\|^2. \tag{10.42}$$

This is the second desired result (10.39), and completes the proof of Corollary 10.3. ∎

Corollary 10.3 implies that for $\|\mathbf{b}\| = O(1)$,

$$\|\mathbf{x}_k\| = O(\frac{1}{\lambda}), \qquad \|\mathbf{x}_\lambda\| = O(\frac{1}{\lambda}), \tag{10.43}$$

where $\lambda \gg \sigma_{min}$, the solution norms of $\mathbf{x}_k$ and $\mathbf{x}_\lambda$ in (10.43) may be reduced significantly, compared with $\mathbf{x}_0$.

Theorem 10.2 *Let (9.10) hold. There exists the bound*

$$\frac{\text{Cond}}{\text{Cond}_k} \geq \frac{\lambda}{\sigma_{min}}. \tag{10.44}$$

For the Tikhonov regularization when $\lambda \leq \sqrt{\sigma_{min}\sigma_{max}}$

$$\frac{\text{Cond}}{\text{Cond}_\lambda} = \frac{\lambda}{\sigma_{min}} \left[\frac{2}{1 + \frac{1}{\text{Cond}} \left(\frac{\lambda^2}{\sigma_{min}\sigma_{max}} \right)} \right] , \tag{10.45}$$

and when $\lambda \geq \sqrt{\sigma_{min}\sigma_{max}}$

$$\frac{\text{Cond}}{\text{Cond}_\lambda} = \frac{\lambda}{\sigma_{min}} \left[\frac{2}{\frac{1}{\text{Cond}} + \frac{\lambda^2}{\sigma_{min}\sigma_{max}}} \right] . \tag{10.46}$$

Proof : We have from $\sigma_k \geq \lambda$,

$$\frac{\text{Cond}}{\text{Cond}_k} = \frac{(\sigma_{max}/\sigma_{min})}{(\sigma_{max}/\sigma_k)} = \frac{\sigma_k}{\sigma_{min}} \geq \frac{\lambda}{\sigma_{min}} . \tag{10.47}$$

This is the first result (10.44).

Next consider $\frac{\text{Cond}}{\text{Cond}_\lambda}$. When $\lambda \leq \sqrt{\sigma_{min}\sigma_{max}}$, we have from Theorem 10.1 and $\lambda \leq \sigma_{max}$.

$$\frac{\text{Cond}}{\text{Cond}_\lambda} = \frac{\sigma_{max}/\sigma_{min}}{(\sigma_{max}^2 + \lambda^2) \,/\, (2\,\lambda\sigma_{max})} = \frac{\lambda}{\sigma_{min}} \left[\frac{2}{1 + \left(\frac{\lambda}{\sigma_{max}} \right)^2} \right] . \tag{10.48}$$

Also, we rewrite (10.48) as

$$\frac{\text{Cond}}{\text{Cond}_\lambda} = \frac{\lambda}{\sigma_{min}} \left[\frac{2}{1 + \frac{\sigma_{min}}{\sigma_{max}} \left(\frac{\lambda^2}{\sigma_{min}\sigma_{max}} \right)} \right] = \frac{\lambda}{\sigma_{min}} \left[\frac{2}{1 + \frac{1}{\text{Cond}} \left(\frac{\lambda^2}{\sigma_{min}\sigma_{max}} \right)} \right] . \tag{10.49}$$

The equations (10.48) and (10.49) are the second desired result (10.45).

Third, when $\lambda \geq \sqrt{\sigma_{min}\sigma_{max}}$, we have from Theorem 10.1 and

$$\begin{aligned} \frac{\text{Cond}}{\text{Cond}_\lambda} &= \frac{\sigma_{max}/\sigma_{min}}{(\sigma_{min}^2 + \lambda^2) \,/\, (2\,\lambda\,\sigma_{min})} = \frac{\lambda}{\sigma_{min}} \left[\frac{2}{\frac{\sigma_{min}}{\sigma_{max}} + \frac{\lambda^2}{\sigma_{min}\sigma_{max}}} \right] \\ &= \frac{\lambda}{\sigma_{min}} \left[\frac{2}{\frac{1}{\text{Cond}} + \frac{\lambda^2}{\sigma_{min}\sigma_{max}}} \right] . \end{aligned} \tag{10.50}$$

This is the third desired result (10.46), and completes the proof of Theorem 10.2. ∎

Corollary 10.4 *Let (9.10) hold. Also assume that*

$$\lambda = \sqrt{\sigma_{min}\sigma_{max}}, \qquad \sigma_{min} \ll \sigma_{max}\,. \tag{10.51}$$

There exist the bounds

$$\frac{\text{Cond}}{\text{Cond}_k} \approx \sqrt{\text{Cond}}\,, \tag{10.52}$$

and

$$\frac{\text{Cond}}{\text{Cond}_\lambda} \approx 2\sqrt{\text{Cond}}\,. \tag{10.53}$$

Proof : We only show (10.53). When (10.51) hold, we have from Theorem 10.2

$$\frac{\text{Cond}}{\text{Cond}_\lambda} = \frac{\lambda}{\sigma_{min}}\left(\frac{2}{1+\frac{1}{\text{Cond}}}\right) = \sqrt{\frac{\sigma_{max}}{\sigma_{min}}}\left(\frac{2}{1+\frac{1}{\text{Cond}}}\right) \approx 2\sqrt{\text{Cond}}\,. \tag{10.54}$$

Corollary 10.4 displays an advantage of Cond_λ over Cond since Cond $\ll 1$. ∎

Theorem 10.3 *Let $\sigma_{min} \ll \sigma_{max}$ and $\lambda = \sigma_k$ when $\lambda \le \sqrt{\sigma_{min}\sigma_{max}}$, there exists the bound,*

$$\text{Cond}_\lambda \approx \frac{1}{2}\text{Cond}_k\,, \tag{10.55}$$

when $\lambda \ge \sqrt{\sigma_{min}\sigma_{max}}$, there exist the bound,

$$\text{Cond}_\lambda \le \text{Cond}_k \quad \text{if} \quad \lambda \le \sqrt{2\sigma_{min}\sigma_{max} - \sigma_{min}^2}\,, \tag{10.56}$$

$$\text{Cond}_\lambda \ge \text{Cond}_k \quad \text{if} \quad \lambda \ge \sqrt{2\sigma_{min}\sigma_{max} - \sigma_{min}^2}\,. \tag{10.57}$$

Proof : When $\lambda = \sigma_k \le \sqrt{\sigma_{min}\sigma_{max}}$, we have from (10.1) and (10.27)

$$\text{Cond}_\lambda = \frac{\sigma_{max} + \frac{\lambda^2}{\sigma_{max}}}{2\lambda} = \frac{\sigma_{max} + \frac{\sigma_k^2}{\sigma_{max}}}{2\sigma_k} = \frac{\sigma_{max}}{2\sigma_k} + \frac{\sigma_k}{2\sigma_{max}} \approx \frac{1}{2}\text{Cond}_k\,, \tag{10.58}$$

by noting that

$$\sigma_k \;=\; \lambda \;\leq\; \sqrt{\sigma_{min}\sigma_{max}} \ll \sigma_{max}\,.$$

Next, when $\lambda = \sigma_k \geq \sqrt{\sigma_{min}\sigma_{max}}$, we have

$$\begin{aligned} \text{Cond}_k - \text{Cond}_\lambda \;&=\; \frac{\sigma_{max}}{\sigma_k} - \frac{\sigma_{min}^2 + \lambda^2}{2\lambda\sigma_{min}} = \frac{\sigma_{max}}{\sigma_k} - \frac{\sigma_{min}^2 + \sigma_k^2}{2\sigma_k\sigma_{min}} \\ &=\; \frac{1}{2\sigma_k\sigma_{min}}[2\sigma_{max}\sigma_{min} - \sigma_{min}^2 - \sigma_k^2]\,. \end{aligned} \tag{10.59}$$

Hence when

$$\sigma_k \;=\; \lambda \leq \sqrt{2\sigma_{max}\sigma_{min} - \sigma_{min}^2}\,,$$

Eq. (10.59) leads to

$$\text{Cond}_k - \text{Cond}_\lambda \geq 0\,,$$

i.e.,

$$\text{Cond}_\lambda \leq \text{Cond}_k\,.$$

This is (10.56). Also when $\sigma_k = \lambda \geq \sqrt{2\sigma_{max}\sigma_{min} - \sigma_{min}^2}$, Eq (10.59) leads to $\text{Cond}_k - \text{Cond}_\lambda \leq 0$, i.e., $\text{Cond}_\lambda \geq \text{Cond}_k$. This is (10.57), and completes the proof of Theorem 10.3. ∎

11 Error Analysis

First, consider the relative errors

$$\delta_k \;=\; \frac{\|\mathbf{x}_0 - \mathbf{x}_k\|}{\|\mathbf{x}_0\|}\,, \tag{11.1}$$

for the TSVD. Denote

$$\mathbf{x}_r \;=\; \mathbf{x}_0 - \mathbf{x}_k \;=\; \sum_{i=k+1}^{n} \frac{\beta_i}{\sigma_i}\mathbf{v}_i\,. \tag{11.2}$$

Lemma 11.1 *Let $k < n$ and*

$$\delta_k = \frac{\|\mathbf{x}_r\|}{\|\mathbf{x}_0\|} \le \varepsilon < 1. \tag{11.3}$$

The necessary condition of (11.3) is

$$\beta_i \le \varepsilon \, \sigma_i \, \|\mathbf{x}_0\| \, , \quad i \ge k. \tag{11.4}$$

Proof : We have from (11.2)

$$\|\mathbf{x}_r\| = \left\{ \sum_{i=k+1}^{n} \left(\frac{\beta_i}{\sigma_i} \right)^2 \right\}^{\frac{1}{2}} . \tag{11.5}$$

Hence, we obtain from (11.3)

$$\frac{\beta_i}{\sigma_i} \le \|\mathbf{x}_r\| \le \varepsilon \|\mathbf{x}_0\| \, , \quad i = k+1, \cdots, n \, . \tag{11.6}$$

This gives the desired result (11.4). ∎

When $\beta_i = u_i^T \mathbf{b} \, (i > k)$ is not small, to satisfy (11.4), the relative errors of $\mathbf{x}_k$ by TSVD may not be small. Hence, the TSVD may not be valid for the problems with high accurate solutions by the collocation Trefftz method (CTM).

Next, we consider the relative errors

$$\delta_\lambda = \frac{\|\mathbf{x}_0 - \mathbf{x}_\lambda\|}{\|\mathbf{x}_0\|} \tag{11.7}$$

for the Tikhonov regularization. We have the following theorem.

Theorem 11.1 *Let (9.10) hold. There exists the bound for the Tikhonov regularization*

$$\frac{\lambda^2}{\sigma_{max}^2 + \lambda^2} \le \frac{\|\mathbf{x}_0 - \mathbf{x}_\lambda\|}{\|\mathbf{x}_0\|} \le \frac{\lambda^2}{\sigma_{min}^2 + \lambda^2} \, . \tag{11.8}$$

Proof : From (9.9) we have

$$\mathbf{x}_0 - \mathbf{x}_\lambda = \sum_{i=1}^{n} \left(\frac{1}{\sigma_i} - \frac{\sigma_i}{\sigma_i^2 + \lambda^2} \right) \beta_i \, \mathbf{v}_i = \sum_{i=1}^{n} \left(\frac{\lambda^2}{\sigma_i^2 + \lambda^2} \right) \frac{\beta_i}{\sigma_i} \, \mathbf{v}_i \, . \tag{11.9}$$

Hence there exists the bound

$$\begin{aligned} \|\mathbf{x}_0 - \mathbf{x}_\lambda\|_0^2 &= \sum_{i=1}^{n} \left(\frac{\lambda^2}{\sigma_i^2 + \lambda^2} \right)^2 \left(\frac{\beta_i}{\sigma_i} \right)^2 \le \max_i \left(\frac{\lambda^2}{\sigma_i^2 + \lambda^2} \right)^2 \sum_{i=1}^{n} \left(\frac{\beta_i}{\sigma_i} \right)^2 \\ &= \left(\frac{\lambda^2}{\sigma_{min}^2 + \lambda^2} \right)^2 \|\mathbf{x}_0\|^2 , \end{aligned} \tag{11.10}$$

to give the bound of the right and side of (11.8). Next from (11.9)

$$\|\mathbf{x}_0 - \mathbf{x}_\lambda\|_0^2 \ge \min_i \left(\frac{\lambda^2}{\sigma_i^2 + \lambda^2} \right)^2 \sum_{i=1}^{n} \left(\frac{\beta_i}{\sigma_i} \right)^2 = \left(\frac{\lambda^2}{\sigma_{max}^2 + \lambda^2} \right)^2 \|\mathbf{x}_0\|^2 , \tag{11.11}$$

to give the bound result of the left hand side of (11.8). This completes the proof of Theorem 11.1. ∎

Theorem 11.2

$$\frac{\lambda^2}{\sigma_{max}^2 + \lambda^2} \le \frac{\|\mathbf{A}\mathbf{x}_k - \mathbf{b}\|}{\|\mathbf{b}\|} \le \frac{\lambda^2}{\sigma_{min}^2 + \lambda^2} \, . \tag{11.12}$$

Corollary 11.1 *Let*

$$\lambda = \sqrt{\sigma_{min}\sigma_{max}} \qquad \sigma_{min} \ll \sigma_{max} \, , \tag{11.13}$$

there exist the bounds

$$\frac{1}{\text{Cond}} \approx c_0 \le \frac{\|\mathbf{x}_0 - \mathbf{x}_\lambda\|}{\|\mathbf{x}_0\|} < 1 \, , \tag{11.14}$$

where Cond $= \frac{\sigma_{max}}{\sigma_{min}}$.

Proof : From the assumptive (11.13), we have

$$\frac{\lambda^2}{\sigma_{min}^2+\lambda^2} = \frac{\sigma_{max}}{\sigma_{min}+\sigma_{max}} < 1 \tag{11.15}$$

and

$$\frac{\lambda^2}{\sigma_{max}^2+\lambda^2} = \frac{\sigma_{min}}{\sigma_{min}+\sigma_{max}} = \frac{1}{\frac{\sigma_{max}}{\sigma_{min}}+1} \approx \frac{1}{\left(\frac{\sigma_{max}}{\sigma_{min}}\right)} = \frac{1}{\text{Cond}}\,.$$

The desired result (11.14) follows from Theorem 11.1. This completes the proof of Corollary 11.1. ∎

Let $\mathbf{b}=\sum_{i=1}^m \beta_i \mathbf{u}_i$, where $\beta_i=\mathbf{u}_i^T\mathbf{b}$. Denote

$$\mathbf{b}_0=\sum_{i=1}^n \beta_i \mathbf{u}_i, \quad \hat{\mathbf{b}}=\sum_{i=n+1}^m \beta_i \mathbf{u}_i\,. \tag{11.16}$$

Then we have

$$\|\mathbf{b}\|=\sqrt{\sum_{i=1}^m \beta_i^2}, \quad \|\mathbf{b}_0\|=\sqrt{\sum_{i=1}^n \beta_i^2}, \quad \|\hat{\mathbf{b}}\|=\sqrt{\sum_{i=n+1}^m \beta_i^2}\,. \tag{11.17}$$

and

$$\|\mathbf{b}\|^2 = \|\mathbf{b}_0\|^2 + \|\hat{\mathbf{b}}\|^2\,. \tag{11.18}$$

We have the following theorem.

Theorem 11.3 *Let (9.10) hold. There exist the bounds*

$$\frac{\lambda^2}{\sigma_{max}^2+\lambda^2} \le \frac{\|\mathbf{b}_0-\mathbf{A}\mathbf{x}_\lambda\|}{\|\mathbf{b}_0\|} \le \frac{\lambda^2}{\sigma_{min}^2+\lambda^2}\,. \tag{11.19}$$

Proof : We have

$$\begin{aligned}\|\mathbf{A}\mathbf{x}_\lambda-\mathbf{b}_0\|_2^2 &= \left\|\mathbf{U}^T\mathbf{A}\mathbf{V}\mathbf{V}^T\mathbf{x}_\lambda-\mathbf{U}^T\mathbf{b}_0\right\| = \left\|\sum_{i=1}^n\left(\frac{\sigma_i^2}{\sigma_i^2+\lambda^2}-1\right)\beta_i\mathbf{u}_i\right\|^2 \\ &= \left\|\sum_{i=1}^n\left(\frac{\lambda^2}{\sigma_i^2+\lambda^2}\right)\beta_i\mathbf{u}_i\right\|^2 = \sum_{i=1}^n\left(\frac{\lambda^2}{\sigma_i^2+\lambda^2}\right)^2\beta_i^2 \end{aligned} \tag{11.20}$$

Then

$$\left(\frac{\lambda^2}{\sigma_{max}^2+\lambda^2}\right)^2 \sum_{i=1}^{n} \beta_i^2 \le \sum_{i=1}^{n} \left(\frac{\lambda^2}{\sigma_i^2+\lambda^2}\right)^2 \beta_i^2 \le \left(\frac{\lambda^2}{\sigma_{\min}^2+\lambda^2}\right)^2 \sum_{i=1}^{n} \beta_i^2 \,. \tag{11.21}$$

The desired results (11.19) follow from (11.16).

Again, the errors $\|\mathbf{A}\mathbf{x}_\lambda - \mathbf{b}\|$ by the Tikhonov regularization may not be small, either. For the problems with noise data, the exact solutions may not exist, or they are meaningless even if existing. The useful solutions such as image processing may have a certain range of errors, which may not be very small, thought. More analysis is given in Hansen [24].

Remark 11.1 *From (11.20), we have*

$$\|\mathbf{A}\mathbf{x}_\lambda - \mathbf{b}\|^2 = \|\hat{\mathbf{b}}_0\|^2 + \|\mathbf{b}_0 - \mathbf{A}\mathbf{x}_\lambda\|^2 = \|\hat{\mathbf{b}}_0\|^2 + \sum_{i=1}^{n} \left(\frac{\lambda^2}{\sigma_i^2+\lambda^2}\right)^2 \beta_i^2 \,.$$

When $\lambda \longrightarrow 0$*,*

$$\|\mathbf{b}_0 - \mathbf{A}\mathbf{x}_\lambda\|^2 = \sum_{i=1}^{n} \left(\frac{\lambda^2}{\sigma_i^2+\lambda^2}\right)^2 \beta_i^2 \longrightarrow 0 \,,$$

and when $\lambda \longrightarrow \infty$*,*

$$\|\mathbf{b}_0 - \mathbf{A}\mathbf{x}_\lambda\|^2 \longrightarrow \sum_{i=1}^{n} \beta_i^2 = \|\mathbf{b}_0\|^2 \,.$$

Moreover, from (10.3) when $\lambda < \sigma_{min}$*, the minimal singular value* $\mu_{min} = O(\sigma_{min})$*. From both errors and ill-conditioning, the assumption (9.10) for* λ *is reasonable.*

12 Combinations of the TSVD and Tikhonov regularization

In this section we consider the computational formula (9.11), where λ is chosen to satisfy

$$\sigma_k \le \lambda < \sigma_{max} \,, \quad \sigma_k \ll \sigma_{max} \,. \tag{12.1}$$

Replacing σ_{min} by σ_k, we have the following lemma from Lemmas 10.1 and 10.2.

Lemma 12.1 *Let (12.1) hold, for the solution by the combination of truncated, Tikhonov regularization (9.11), the singular values of (10.3) have the following bounds,*

$$\min_{k\leq i\leq n} \mu_i \;=\; 2\lambda\,, \tag{12.2}$$

$$\max_{k\leq i\leq n} \mu_i \;=\; \sigma_{max} + \frac{\lambda^2}{\sigma_{max}} \quad \textit{if } \lambda \leq \sqrt{\sigma_k\sigma_{max}}\,, \tag{12.3}$$

$$\max_{k\leq i\leq n} \mu_i \;=\; \sigma_k + \frac{\lambda^2}{\sigma_k} \quad \textit{if } \lambda \geq \sqrt{\sigma_k\sigma_{max}}\,. \tag{12.4}$$

Theorem 12.1 *Let (12.1) hold. The condition number and the effective condition number for the truncated Tikhonov regularization are given by*

$$\text{Cond\_eff}_{k-\lambda} \;=\; \frac{\|\mathbf{b}\|}{2\lambda\|\mathbf{x}_{k-\lambda}\|}\,, \tag{12.5}$$

$$\text{Cond}_{k-\lambda} \;=\; \frac{\sigma_{max}^2+\lambda^2}{2\lambda\,\sigma_{max}} \quad \textit{if} \;\; \lambda \leq \sqrt{\sigma_k\sigma_{max}}\,, \tag{12.6}$$

$$\text{Cond}_{k-\lambda} = \frac{\sigma_k^2+\lambda^2}{2\lambda\,\sigma_k} \quad \textit{if} \;\; \lambda \geq \sqrt{\sigma_k\sigma_{max}}\,. \tag{12.7}$$

We have the following corollary.

Corollary 12.1 *There exists the bound*

$$\text{Cond}_{k-\lambda} \leq \text{Cond}_{\lambda}\,. \tag{12.8}$$

Proof : If

$$\lambda \leq \sqrt{\sigma_{min}\sigma_{max}} \leq \sqrt{\sigma_k\sigma_{max}}\,, \tag{12.9}$$

from (10.27) and (12.6) we have $\text{Cond}_{k-\lambda} \;=\; \text{Cond}_{\lambda}$. Next, if

$$\lambda \geq \sqrt{\sigma_{min}\sigma_{max}} \geq \sqrt{\sigma_k\sigma_{min}}\,, \tag{12.10}$$

for function (10.10) in $\sigma_{min} < y < \sigma_k$,

$$f^{'}(y) \;=\; 1 - \frac{\lambda^2}{y^2} < 0\,, \tag{12.11}$$

to indicate

$$f(\sigma_{min}) > f(\sigma_k)\,, \tag{12.12}$$

i.e.,

$$\sigma_{min} + \frac{\lambda^2}{\sigma_{min}} > \sigma_k + \frac{\lambda^2}{\sigma_k}\,. \tag{12.13}$$

Hence we have

$$\mathrm{Cond}_\lambda \;=\; \frac{\sigma_{min} + \frac{\lambda^2}{\sigma_{min}}}{2\lambda} \geq \frac{\sigma_k + \frac{\lambda^2}{\sigma_k}}{2\lambda} \;=\; \mathrm{Cond}_{k-\lambda}\,.\; \blacksquare \tag{12.14}$$

Finally, consider the case:

$$\sqrt{\sigma_{min}\sigma_{max}} \leq \lambda \leq \sqrt{\sigma_k\sigma_{max}}\,, \tag{12.15}$$

to prove (12.8), which can be written by noting (10.28) and (12.6)

$$\frac{\sigma_{max}^2 + \lambda^2}{2\lambda\sigma_{max}} \leq \frac{\sigma_{min}^2 + \lambda^2}{2\lambda\sigma_{min}}\,. \tag{12.16}$$

The equation (12.16) holds due to Theorem 10.1 under (12.15).

Corollary 12.2 *Let $k \leq n$ and $\sigma_k \leq \lambda \leq \sigma_{max}$, there exists the bound*

$$\mathrm{Cond\_eff}_\lambda \leq \mathrm{Cond\_eff}_{k-\lambda}\,. \tag{12.17}$$

Proof : We have

$$\sum_{i=1}^{k} \left(\frac{\beta_i}{\sigma_i + \frac{\lambda^2}{\sigma_i}} \right)^2 \leq \sum_{i=1}^{n} \left(\frac{\beta_i}{\sigma_i + \frac{\lambda^2}{\sigma_i}} \right)^2 ,$$

i.e.,

$$\|\mathbf{x}_{k-\lambda}\| \le \|\mathbf{x}_\lambda\| \,.$$

Hence we obtain

$$\text{Cond\_eff}_\lambda = \frac{\|\mathbf{b}\|}{2\lambda\|\mathbf{x}_\lambda\|} \le \frac{\|\mathbf{b}\|}{2\lambda\|\mathbf{x}_{k-\lambda}\|} \;=\; \text{Cond\_eff}_{k-\lambda} \,.$$

This is the desired result (12.17), and completes the proof of Corollary 12.2. ∎

Theorem 12.2 *Let $\sigma_k \le \lambda \le \sigma_{max}$ and $\mathbf{x}_r \;=\; \mathbf{x}_0 - \mathbf{x}_k$ satisfy*

$$\frac{\|\mathbf{x}_r\|}{\|\mathbf{x}_0\|} \;=\; \varepsilon < 1 \,. \tag{12.18}$$

There exists the bound

$$\frac{\|\mathbf{x}_0 - \mathbf{x}_{k-\lambda}\|}{\|\mathbf{x}_0\|} \le \varepsilon + \frac{\lambda^2}{\sigma_k^2 + \lambda^2}\sqrt{1-\varepsilon^2} \,. \tag{12.19}$$

Proof : From (12.18) and

$$\|\mathbf{x}_k\|^2 \;\le\; \|\mathbf{x}_0\|^2 - \|\mathbf{x}_r\|^2 \;=\; (1-\varepsilon^2)\|\mathbf{x}_0\|^2 \,,$$

we have

$$\|\mathbf{x}_k\| \;\le\; \sqrt{1-\varepsilon^2}\|\mathbf{x}_0\| \,. \tag{12.20}$$

From the triangle inequality

$$\|\mathbf{x}_0 - \mathbf{x}_{k-\lambda}\| \le \|\mathbf{x}_0 - \mathbf{x}_k\| + \|\mathbf{x}_k - \mathbf{x}_{k-\lambda}\| \le \|\mathbf{x}_r\| + \|\mathbf{x}_k - \mathbf{x}_{k-\lambda}\| \,, \tag{12.21}$$

we have from (9.5) and (9.11),

$$\mathbf{x}_k - \mathbf{x}_{k-\lambda} \;=\; \sum_{i=1}^{k} \frac{\lambda^2}{\sigma_i^2 + \lambda^2}\frac{\beta_i}{\sigma_i}\mathbf{v}_i \tag{12.22}$$

to give

$$\begin{aligned}
\|\mathbf{x}_k - \mathbf{x}_{k-\lambda}\|^2 &= \sum_{i=1}^{k} \left(\frac{\lambda^2}{\sigma_i^2+\lambda^2}\right)^2 \left(\frac{\beta_i}{\sigma_i}\right)^2 \le \left(\frac{\lambda^2}{\sigma_k^2+\lambda^2}\right)^2 \sum_{i=1}^{k} \left(\frac{\beta_i}{\sigma_i}\right)^2 \\
&= \left(\frac{\lambda^2}{\sigma_k^2+\lambda^2}\right)^2 \|\mathbf{x}_k\|^2 . \qquad (12.23)
\end{aligned}$$

Hence we have for (12.23) and (12.18)

$$\|\mathbf{x}_k - \mathbf{x}_{k-\lambda}\|^2 \le \left(\frac{\lambda^2}{\sigma_k^2+\lambda^2}\right)^2 \|\mathbf{x}_k\|^2 \le \left(\frac{\lambda^2}{\sigma_k^2+\lambda^2}\right)^2 (1-\varepsilon^2)\|\mathbf{x}_0\|^2 . \qquad (12.24)$$

Combining (12.21) and (12.24) gives

$$\|\mathbf{x}_0 - \mathbf{x}_{k-\lambda}\| \le \left[\varepsilon + \left(\frac{\lambda^2}{\sigma_k^2+\lambda^2}\right)\sqrt{1-\varepsilon^2}\right]\|\mathbf{x}_0\| . \qquad (12.25)$$

This is the deserved result (12.19), and completes the proof of Theorem 12.2.

Corollary 12.3 *Let (12.18) hold $\lambda = \sigma_k$, there exists the bound*

$$\frac{\|\mathbf{x}_0 - \mathbf{x}_{k-\lambda}\|}{\|\mathbf{x}_0\|} \le \varepsilon + \frac{1}{2} . \qquad (12.26)$$

Moreover, when $\lambda \gg \sigma_k$, Eq. (12.19) leads to

$$\frac{\|\mathbf{x}_0 - \mathbf{x}_{k-\lambda}\|}{\|\mathbf{x}_0\|} \approx 1 . \qquad (12.27)$$

This implies that λ should not be chosen too large as $\lambda \ge \sqrt{\sigma_{min}\sigma_{max}}$.

Remark 12.1 *The truncated Tikhonov regularization is, indeed, the Tikhonov regularization for the rank deficient with $\sigma_{k+1} = \cdots = \sigma_n = 0$.*

13 Choice of Parameter λ in the Tikhonov Regularization

In this section, we consider how to choose the parameter λ in the Tikhonov regularization. In Hansen [25], and Hansen and O'Leary [26], the L-curve is used, where the parametric functions

$$\|\mathbf{x}_\lambda\| = y = y(\lambda)\,, \tag{13.1}$$
$$\|\mathbf{A}\mathbf{x}_\lambda - \mathbf{b}\| = \mathbf{x} = \mathbf{x}(\lambda)\,. \tag{13.2}$$

Denote a curve $y = y(\mathbf{x})$ on the Cartesian coordinate system XOY. The curve looks as the L-shaped, and called the L-curve. A batter choice of λ is at the corner of the L-curve. Other study on the choice of parameter λ is reported in Neumaier [44], Engle and Gfrerer [19, 20] and Chan et al. [6].

In this section, we first give some analysis for choice of λ by the curve of $(x, y) = (\|\mathbf{A}\mathbf{x}_\lambda - \mathbf{b}\|, \|\mathbf{x}_\lambda\|)$, and then explore that by the other curves of $(x, y) = (\|\mathbf{A}\mathbf{x}_\lambda - \mathbf{b}\|, \text{Cond}_\lambda)$ and $(x, y) = (\|\mathbf{A}\mathbf{x}_\lambda - \mathbf{b}\|, \text{Cond\_eff}_\lambda)$. From (9.7), under a given λ, we have

$$\|\mathbf{A}\mathbf{x}_k - \mathbf{b}\|^2 + \lambda^2\|\mathbf{x}_k\|^2 = \min_{\mathbf{x}\in R^n}(\|\mathbf{A}\mathbf{x}_\lambda - \mathbf{b}\| + \lambda^2\|\mathbf{x}\|^2)\,. \tag{13.3}$$

Denote the energy of λ,

$$T(\lambda) = \|\mathbf{A}\mathbf{x}_\lambda - \mathbf{b}\|^2 + \lambda^2\|\mathbf{x}_\lambda\|^2\,. \tag{13.4}$$

Then we have from (11.20)

$$\|\mathbf{A}\mathbf{x}_\lambda - \mathbf{b}\|^2 = \|\hat{\mathbf{b}}\|^2 + \sum_{i=1}^{n}\left(\frac{\lambda^2}{\sigma_i^2 + \lambda^2}\right)^2\beta_i^2 \tag{13.5}$$

and

$$\|\mathbf{x}_\lambda\|^2 = \sum_{i=1}^{n}\left(\frac{\sigma_i^2}{\sigma_i^2 + \lambda^2}\right)^2\beta_i^2\,. \tag{13.6}$$

Hence we have

$$\begin{aligned} T(\lambda) &= \|\hat{\mathbf{b}}\|^2 + \sum_{i=1}^{n}\left(\frac{\lambda^2}{\sigma_i^2 + \lambda^2}\right)^2\beta_i^2 + \lambda^2\sum_{i=1}^{n}\left(\frac{\sigma_i^2}{\sigma_i^2 + \lambda^2}\right)^2\beta_i^2 \\ &= \|\hat{\mathbf{b}}\|^2 + \sum_{i=1}^{n}\frac{(\sigma_i^2\lambda^2 + \lambda^4)}{(\sigma_i^2 + \lambda^2)^2}\beta_i^2\,. \end{aligned} \tag{13.7}$$

For simplicity, let $z = \lambda^2$, Eq. (13.7) is written as

$$f(z) = T(\lambda) = \|\hat{\mathbf{b}}\|^2 + \sum_{i=1}^{n} \frac{(\sigma_i^2 \, z + z^2)}{(\sigma_i^2 + z)^2} \beta_i^2 \,. \tag{13.8}$$

We then have the following theorem.

Theorem 13.1 *For the energy (13.8), there exist the inequalities*

$$\frac{\partial f(z)}{\partial z} \;=\; \|\mathbf{x}_\lambda\|^2 > 0 \,, \tag{13.9}$$

$$\frac{\partial^2 f(z)}{\partial z^2} \;<\; 0 \,. \tag{13.10}$$

Proof : By calculus, we have

$$\frac{d}{dz}\left(\frac{\sigma_i^2 \, z + z^2}{(\sigma_i^2 + z)^2}\right) = \frac{\sigma_i^2}{(\sigma_i^2 + z)^2} \,. \tag{13.11}$$

Hence from (13.8) and (13.11)

$$f'(z) = \sum_{i=1}^{n} \beta_i^2 \frac{d}{dz}\left[\frac{\sigma_i^2 z + z^2}{(\sigma_i^2 + z)^2}\right] = \sum_{i=1}^{n} \frac{\beta_i^2 \sigma_i^2}{(\sigma_i^2 + z)^2} = \sum_{i=1}^{n} \frac{\sigma_i^2}{(\sigma_i^2 + \lambda^2)^2} \beta_i^2 = \|\mathbf{x}_\lambda\|^2 \,. \tag{13.12}$$

This is the first result (13.9). Next we have

$$f''(z) = \frac{d}{dz}\left(\|\mathbf{x}_\lambda\|^2\right) = \sum_{i=1}^{n} \sigma_i^2 \beta_i^2 \frac{d}{dz}\left(\frac{1}{(\sigma_i^2 + z)^2}\right) = -\sum_{i=1}^{n} \sigma_i^2 \beta_i \frac{1}{(\sigma_i^2 + z)^3} < 0 \,. \tag{13.13}$$

This completes the proof of Theorem 13.1. ∎

The energy $f(z) = f(\lambda^2)$ in (13.8) can be drawn in Figure 7, and the concave curve of $f(z)$ monotonic increases with respect to $z(= \lambda^2)$. Hence, the minimum of $f(z)$ is given by $z = \sigma_{min}^2$.

Below let us retain the errors $\|\mathbf{b} - \mathbf{A}\mathbf{x}_k\|$, but replace $\|\mathbf{x}_\lambda\|$ by Cond_λ and Cond_eff_λ, which indicate directly the ill-conditioning. When $\|\mathbf{b}\| = O(\sigma_{max})$, we have from (10.5) and (10.6),

$$\text{Cond}_\lambda \approx O(\text{Cond\_eff}_\lambda \|\mathbf{x}_\lambda\|) \,.$$

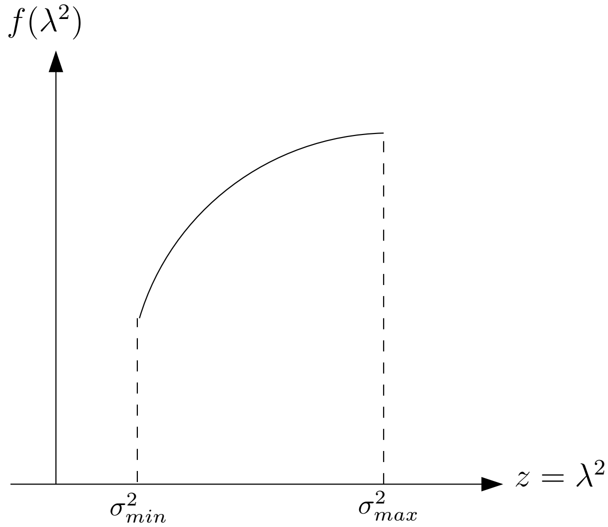

Figure 7: The curve of $f(z)$.

Hence Cond_λ can be considered as the global ill-conditioning of both $\mathbf{x}_\lambda$ and the final solution (9.11).

We have the following lemma.

Lemma 13.1 *Let (9.10) hold. There exist the inequalities.*

$$\frac{d}{d\lambda}(\text{Cond}_\lambda) < 0\,, \quad \frac{d^2}{d\lambda^2}(\text{Cond}_\lambda) > 0\,, \quad \text{if} \ \ \lambda \leq \sqrt{\sigma_{min}\sigma_{max}}\,, \tag{13.14}$$

and

$$\frac{d}{d\lambda}(\text{Cond}_\lambda) > 0\,, \quad \frac{d^2}{d\lambda^2}(\text{Cond}_\lambda) > 0\,, \quad \text{if} \ \ \lambda \geq \sqrt{\sigma_{min}\sigma_{max}}\,. \tag{13.15}$$

Proof : When $\lambda \leq \sqrt{\sigma_{min}\sigma_{max}}$, we have from (10.27)

$$\text{Cond}_\lambda = \frac{\sigma^2_{max} + \lambda^2}{2\lambda\sigma_{max}}\,. \tag{13.16}$$

Then, under (9.10) there exist the inequalities,

$$\frac{d}{d\lambda}(\text{Cond}_\lambda) = \frac{1}{2\,\sigma_{max}}\left[1 - \frac{\sigma^2_{max}}{\lambda^2}\right] < 0\,, \tag{13.17}$$

and

$$\frac{d^2}{d\lambda^2}(\mathrm{Cond}_\lambda) = \frac{\sigma_{max}}{\lambda^3} > 0\,. \tag{13.18}$$

This is the first results (13.14). Next, from (10.28) we have

$$\mathrm{Cond}_\lambda = \frac{\sigma_{min}^2 + \lambda^2}{2\lambda\,\sigma_{min}}\,. \tag{13.19}$$

Then under (9.10), there exist the inequalities

$$\frac{d}{d\lambda}(\mathrm{Cond}_\lambda) = \frac{1}{2\sigma_{min}}\left[1 - \frac{\sigma_{min}^2}{\lambda^2}\right] > 0\,, \tag{13.20}$$

and

$$\frac{d^2}{d\lambda^2}(\mathrm{cond}_\lambda) = \frac{\sigma_{min}}{\lambda^3} > 0\,. \tag{13.21}$$

This is the second results (14.13), and completes the proof of Lemma 13.1. ∎

We may draw the curve of Cond_λ in Figure 8, and a minimum of cond_λ is found at $\lambda = \sqrt{\sigma_{min}\sigma_{max}}$.

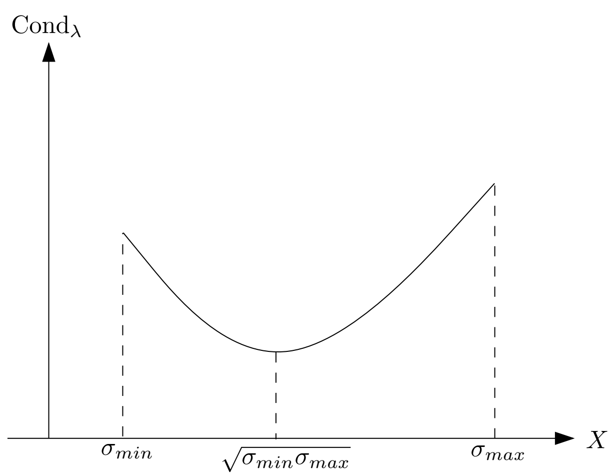

Figure 8: The curve of Cond_λ.

Lemma 13.2 *When* $\lambda = \sqrt{\sigma_{min}\sigma_{max}}$,

$$\text{Cond}_\lambda \approx \frac{1}{2}\sqrt{\text{Cond}}\,, \tag{13.22}$$

and when $\sigma_k = \sqrt{\sigma_{min}\sigma_{max}}$,

$$\text{Cond}_\lambda = \sqrt{\text{Cond}}\,. \tag{13.23}$$

Proof : We have Cond_λ at $\lambda = \sqrt{\sigma_{min}\sigma_{max}}$,

$$\begin{aligned} \text{Cond}_\lambda &= \frac{\sigma_{max}^2 + \lambda^2}{2\lambda\,\sigma_{max}} = \frac{\sigma_{max} + \sigma_{min}}{2\sqrt{\sigma_{min}\sigma_{max}}} = \frac{1}{2}\left(\sqrt{\frac{\sigma_{max}}{\sigma_{min}}} + \sqrt{\frac{\sigma_{min}}{\sigma_{max}}}\right) \\ &= \frac{1}{2}\left(\sqrt{\text{Cond}} + \frac{1}{\sqrt{\text{Cond}}}\right) \approx \frac{1}{2}\sqrt{\text{Cond}}\,. \end{aligned} \tag{13.24}$$

Next, when $\sigma_k = \sqrt{\sigma_{min}\sigma_{max}}$, we have

$$\text{Cond}_k = \frac{\sigma_{max}}{\sigma_k} = \sqrt{\frac{\sigma_{max}}{\sigma_{min}}} = \sqrt{\text{Cond}}\,. \blacksquare \tag{13.25}$$

Lemma 13.3 *For* $\lambda > 0$, *there exists the inequality,*

$$\frac{d}{d\lambda}\left(\|\mathbf{b} - \mathbf{A}\mathbf{x}_\lambda\|^2\right) > 0\,. \tag{13.26}$$

Proof : We have from

$$\|\mathbf{b} - \mathbf{A}\mathbf{x}_\lambda\|^2 = \|\hat{\mathbf{b}}\|^2 + \sum_{i=1}^{n}\left(\frac{\lambda^2}{\sigma_i^2 + \lambda^2}\right)\beta_i^2 = \|\hat{\mathbf{b}}\|^2 + \sum_{i=1}^{n}\frac{1}{1 + \left(\dfrac{\sigma_i}{\lambda}\right)^2}\beta_i^2\,. \tag{13.27}$$

Evidently, when λ increases, $\|\mathbf{b} - \mathbf{A}\mathbf{x}_\lambda\|^2$ increases. This completes the proof of Lemma 13.3. $\blacksquare$

From Lemmas 13.1 and 13.2, when $\lambda \geq \sqrt{\sigma_{min}\sigma_{max}}$, both Cond_λ and $\|\mathbf{b} - \mathbf{A}\mathbf{x}_\lambda\|$ increase. Hence, the optimal λ_{opt} should be chosen as

$$\sigma_{min} \leq \lambda_{opt} \leq \sqrt{\sigma_{min}\sigma_{max}}\,. \tag{13.28}$$

When (13.28) holds, Cond_λ decreases and $\|\mathbf{b} - \mathbf{A}\mathbf{x}_\lambda\|$ increases when λ increases. A balance between $\|\mathbf{A}\mathbf{x}_\lambda - \mathbf{b}\|$ and Cond_λ should be made. Compared with the MFS, the ratios of

$$\alpha_\varepsilon = \frac{\|\mathbf{A}\mathbf{x}_\lambda - \mathbf{b}\|}{\|\mathbf{A}\mathbf{x}_0 - \mathbf{b}\|}, \quad \beta_{\text{Cond}} = \frac{\text{Cond}_\lambda}{\text{Cond}}, \tag{13.29}$$

are more interesting, where $\mathbf{x}_0$ is the solution of MFS without the Tikhonov regularization, and Cond is the traditional condition number. Hence we may draw the curve of $(x, y) = (\alpha_\varepsilon, \beta_{\text{Cond}})$ under (13.28). Similarly, we may also draw the curve of $(x, y) = (\alpha_\varepsilon, \beta_{\text{Cond\_eff}})$, where

$$\beta_{\text{Cond\_eff}} = \frac{\text{Cond\_eff}_\lambda}{\text{Cond\_eff}}. \tag{13.30}$$

Similarly, we may draw the L-curve of $(x, y) = (\alpha_\varepsilon, \beta_{\|\mathbf{x}_k\|})$, where

$$\beta_{\|\mathbf{x}_k\|} = \frac{\|\mathbf{x}_k\|}{\|\mathbf{x}_0\|}. \tag{13.31}$$

Furthermore, we may compute their ratios

$$\frac{y}{x} = \frac{\beta_{\text{Cond}}}{\alpha_\varepsilon}, \quad \frac{\beta_{\text{Cond\_eff}}}{\alpha_\varepsilon}, \quad \frac{\beta_{\|\mathbf{x}_k\|}}{\alpha_\varepsilon},$$

and draw the curves of $(\lambda, \frac{y}{x})$. All those curves may indicate a better choice of parameter λ.

Finally, let us seek the balance solution between α_ε and β_{Cond}. Define an energy

$$I(\lambda, w) = \alpha_\varepsilon^2 + w^2 \beta_{\text{Cond}}^2 = \frac{\|\mathbf{A}\mathbf{x}_k - \mathbf{b}\|^2}{\|\mathbf{A}\mathbf{x}_0 - \mathbf{b}\|^2} + w^2 \left(\frac{\text{Cond}_\lambda}{\text{Cond}} \right)^2, \tag{13.32}$$

where $w(> 0)$ is a suitable weight. For a given w, the optimal parameter λ can be found by

$$I(\lambda_{opt}, w) = \min_{\lambda \in \wedge} I(\lambda, w), \tag{13.33}$$

where $\lambda = [\sigma_{min}, \sqrt{\sigma_{min}\sigma_{max}}]$. For simplicity, let $z = \lambda^2$, we have

$$\begin{aligned} g(z) = I(\lambda, w) &= \frac{1}{\|\mathbf{b} - \mathbf{A}\mathbf{x}_0\|^2} \left\{ \|\mathbf{b}_0\|^2 + \sum_{i=1}^{n} \left[\frac{\sigma_i^2 \, z + z^2}{(\sigma_i^2 + z)^2} \right] \beta_i^2 \right\} \\ &+ w^2 \frac{1}{\text{Cond}^2} \left[\frac{\sigma_{max}^2 + z}{2\sqrt{z}\,\sigma_{max}} \right]^2 . \end{aligned} \tag{13.34}$$

The optimal $z(= \lambda^2)$ can be sought by

$$\frac{d}{dz} g(z) = 0\,, \tag{13.35}$$

under a given w. Since

$$\frac{d}{dz}\left(\frac{\sigma_{max}^2 + z}{2\sqrt{z}\sigma_{max}} \right) = -\frac{1}{4\sigma_{max}} \left(\frac{\sigma_{max}^2}{z} - 1 \right) ,$$

we have from Theorem 13.1,

$$\frac{d}{dz} g(z) = \Phi(z) = -\frac{w^2}{4} \frac{\text{Cond}_\lambda}{\text{Cond}^2} \frac{1}{\sigma_{max}} \left(\frac{\sigma_{max}^2}{z} - 1 \right) + \frac{\|\mathbf{x}_\lambda\|^2}{\|\mathbf{b} - \mathbf{A}\mathbf{x}_0\|^2} = 0\,. \tag{13.36}$$

Hence the potential optimal $\bar{z} \in [\sigma_{min}^2, \sigma_{min}\sigma_{max}]$ satisfies

$$\Phi(\bar{z}) = 0\,. \tag{13.37}$$

We write this result as a theorem.

Theorem 13.2 *Let (9.10) hold. For a given $w(> 0)$, the optimal z_{opt} may be found by*

$$g(z_{opt}) = \min \left\{ g(\sigma_{min}^2), \quad g(\sigma_{min}\sigma_{max}), \quad g(\bar{z}) \right\} ,$$

where $\bar{z}$ is given in (13.37).

In order to obtain $\bar{z}$, we may use the Newton iteration method.

$$z_{k+1} = z_k + \frac{\Phi(z_k)}{\Phi'(z_k)}\,, \quad k = 0, 1, \cdots , \tag{13.38}$$

where z_0 is a good initial value. The good initial value z_0 can be obtained by the interplant, based on the curves of (x, y) given.

For a given λ, the solution $\mathbf{x}_k$ by the Tikhonov regularization is obtained by

$$T(\mathbf{x}_k) = \min_{\mathbf{x} \in R^n} T(\mathbf{x}) \,, \tag{13.39}$$

where

$$T = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 + \lambda^2 \|\mathbf{x}\|^2 \,. \tag{13.40}$$

Hence we have from (13.4)

$$T(\mathbf{x}_k) \;=\; \|\mathbf{A}\mathbf{x}_k - \mathbf{b}\|^2 + \lambda^2 \|\mathbf{x}_k\|^2 = \|\mathbf{A}\mathbf{x}_k - \mathbf{b}\|^2 + \frac{\|\mathbf{b}\|^2}{4\text{Cond\_eff}_k^2} \,. \tag{13.41}$$

Since Eq. (13.39) is, indeed, the least squares method, the two terms in (13.41) should be balanced in small magnitude to each other, i.e.,

$$\|\mathbf{A}\mathbf{x}_k - \mathbf{b}\|^2 \asymp \frac{\|\mathbf{b}\|^2}{\text{Cond\_eff}_k^2} \,. \tag{13.42}$$

This gives a relation

$$\|\mathbf{A}\mathbf{x}_k - \mathbf{b}\| \times \text{Cond\_eff}_k \asymp \|\mathbf{b}\| \,. \tag{13.43}$$

We write this result as a theorem.

Theorem 13.3 *Under a given λ, for the optimal solution $\mathbf{x}_k$ from the Tikhonov regularization, there exists the relation (13.43).*

Theorem 13.3 implies that the products of errors and effective condition numbers (i.e., ill-conditioning) retain globally invariant!

14 Numerical Experiments

Consider the Dirichlet problem of Laplace's equation

$$\Delta u = 0 \text{ in } S\,, \quad u = g \text{ on } \partial S\,, \tag{14.1}$$

where $S = \{(x,y)| -1 < x < 1\,,\ 0 < y < 1\}$. We choose the smooth solution

$$u = \sin k\pi x \sinh k\pi y\,, \quad k = 1, 2. \tag{14.2}$$

The Dirichlet condition is given explicitly by (see Figure 9)

$$u\Big|_{\overline{AB}\cup\overline{CD}\cup\overline{AD}} = 0\,, \tag{14.3}$$

$$u\Big|_{\overline{BC}} = g = \sin k\pi x \sinh k\pi\,. \tag{14.4}$$

Let $G(0, \frac{1}{2})$ be the origin of the polar coordinates (r, θ). Then

$$r_{max} = \max_S r = \overline{GB} = \frac{\sqrt{5}}{2}\,.$$

Choose the resource points Q_i uniformly on the outside circle,

$$Q_i = \{(r,\theta)|\, r = R\,,\ \theta = ih\}\,, \tag{14.5}$$

where $R > r_{max}$, and $h = \frac{2\pi}{N}$. We use the fundamental solutions

$$\phi_i(P) = \ln|\overline{PQ_i}|\,, \quad i = 1, 2, \cdots, N\,, \tag{14.6}$$

where $P \in S \cup \partial S$, and choose their linear combination

$$u_N = \sum_{i=1}^{N} c_i \phi_i(P)\,, \tag{14.7}$$

as the approximate solutions of (14.1), where c_i are the coefficients to be sought, since the functions (14.7) are harmonic, we may use the collocation method for satisfying the Dirichlet boundary conditions (14.3) and (14.4). Hence we have

$$u_N(P_i) = \sum_{i=1}^{N} c_i \phi_i(P_i) = 0\,, \quad P_i \in \overline{AB} \cup \overline{CD} \cup \overline{AD}\,, \tag{14.8}$$

$$u_N(P_i) = \sum_{i=1}^{N} c_i \phi_i(P_i) = g(P_i)\,, \quad P_i \in \overline{BC}\,. \tag{14.9}$$

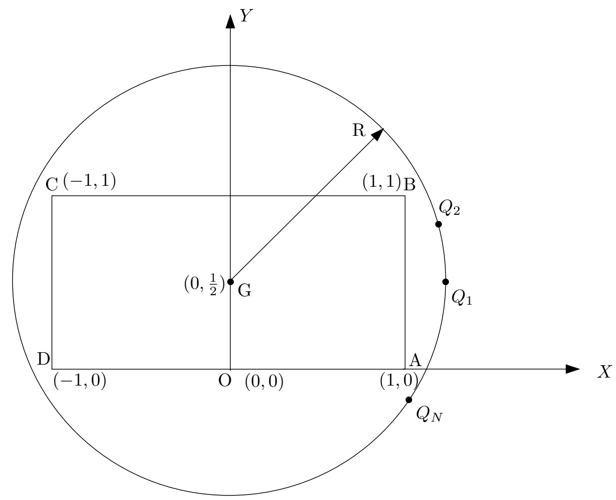

Figure 9: rectangular domain.

For simplicity, we may choose the uniform collocation points P_i. Such approaches can be described as the least squares method: To seek $u_N \in V_L$ such that

$$u_N = \min_{v \in V_L} \hat{I}(v), \tag{14.10}$$

where V_L is the set of (14.7), and

$$\hat{I}(v) = \widehat{\int}_\Gamma (v-g)^2. \tag{14.11}$$

In (14.11), $g = 0$ on $\overline{AB} \cup \overline{CD} \cup \overline{AD}$ and $g = \sin k\pi x \sinh k\pi$ in (14.4) on $\overline{BC}$, and the approximation of $\int_\Gamma$ by $\widehat{\int}_\Gamma$ is the central rule. We may also establish the collocation equations by the Gaussian rule

$$w_i \sum_{i=1}^{N} c_i \phi_i(P_i) = 0, \quad P_i \in \overline{AB} \cup \overline{CD} \cup \overline{AD}, \tag{14.12}$$

$$w_i \sum_{i=1}^{N} c_i \phi_i(P_i) = w_i\, g(P_i), \quad P_i \in \overline{BC}, \tag{14.13}$$

where w_i and P_i are the weights and integration nodes, respectively. Since the fundamental solutions are used, the fundamental solution method (FSM) is called in this thesis.

Let M denote the number of uniform collocation nodes along $\overline{AB}$. Hence the total number of collocation equations is $6M$, see Figure 9. Choose $6M > N$ and $R = \sqrt{3} >$

$\frac{\sqrt{5}}{2}$. The errors and condition numbers are given Table 16, where

$$\|\varepsilon\|_B = \left\{ \int_\Gamma (u_N - g)^2 \right\}^{\frac{1}{2}} . \tag{14.14}$$

For Table 16, we can see

$$\begin{aligned} \|\varepsilon\|_B &= O(0.46^N)\,, \\ \sigma_{max} &= O(1)\,, \qquad\qquad \sigma_{min} = O(0.5^N)\,, \\ \text{Cond} &= O(3.04^N)\,, \quad \text{Cond\_eff} = O(2.01^N)\,, \\ \|\mathbf{x}\| &= O(1.48^N)\,. \end{aligned}$$

The coefficients c_i, and singular values σ_i are listed in Tables 17 and 18 respectively. From Table 17, some coefficients c_i are huge as $O(10^8)$ and oscillating. From Table 18, $\sigma_{max} = 1.34$ and $\sigma_{min} = O(10^{-26})$. The minimal singular value is infinitesimal.

Since σ_{min} is infinitesimal, the coefficients c_i are huge and the ill-conditioning is very severe. To improve the ill-conditioning, we solicit the TSVD and the Tikhonov regularization. First choose $\lambda = \sigma_k$ in the TSVD, for the solution of $N = 71$ and $M = 50$ in Table 17, the errors and condition numbers are listed in Table 19. When $k = 71$ in Table 19, the solution is just that of Table 16 at $N = 71$ and $M = 50$. Evidently, the solution $\mathbf{x}_k$ of coefficients c_i is deduced from $O(10^8)$ to $O(10^2)$ when $k \geq 57$, and Cond_k from $O(10^{24})$ to $O(10^{17})$ and below. Table 20 lists the small coefficients by TSVD. When $k = 57$, the errors $\|\varepsilon\|_B = O(10^{-15})$ increase only by a factor of 10, and the effective condition number $\text{Cond\_eff}_k = O(10^{14})$ is deduced by a factor of 100. Moreover, when $k = 55$, $\sigma_k \approx \sqrt{\sigma_{min}\sigma_{max}}$. Note that the solution errors $\|\varepsilon\|_B$ increase significantly when $k < 55$, i.e., $\sigma_k > \sqrt{\sigma_{min}\sigma_{max}}$. Hence, it is better to choose the small σ_k such that $\sigma_k < \sqrt{\sigma_{min}\sigma_{max}}$, also see Corollary 12.3.

Next, we choose $\lambda = \sigma_m$ in the Tikhonov regularization, the errors and condition numbers are listed in Table 21, and the coefficients in Table 17. When $m = 57$, the errors and condition numbers have the similar behaviors of those by TSVD. When $k = m = 57$,

$$\text{Cond}_\lambda = 8.889(14), \quad \text{Cond}_k = 1.778(15)\,,$$

to give

$$\text{Cond}_\lambda \approx \frac{1}{2}\text{Cond}_k\,. \tag{14.15}$$

This coincides with (10.55), perfectly. The data of Cond_λ and Cond_k in Tables 19 and 21 and affect $k = m > 51$ are also consistent with (10.56) and (10.57).

Finally, for the truncated Tikhonov regularization, we use

$$u_{k-\lambda}^{(I)} = \sum_{i=1}^{k} \frac{\beta_i}{\sigma_i + \dfrac{\sigma_k^2}{\sigma_i}} \mathbf{v}_i \,, \tag{14.16}$$

and

$$u_{k-\lambda}^{(II)} = \sum_{i=1}^{k} \frac{\beta_i}{\sigma_i + \dfrac{\lambda^2}{\sigma_i}} \mathbf{v}_i \,, \quad \lambda = \sigma_m \quad m < k \,. \tag{14.17}$$

Tables 23 and 24 list the results by (14.16) and (14.17) with $m = k - 1$, respectively, and Table 25 lists the results by (14.17) as $k = 55$. It seems that when the differences $k - m > 1$, the errors $\|\varepsilon\|_B$ increase significantly. Hence we choose $k = m + 1$, as in Table 24.

From all Tables, we can see that Eqs (10.7) and (10.8), hold to imply that Cond_eff is always smaller than Cond. From Tables 21 and 23, when $m = 65$

$$\begin{aligned} \text{Cond\_eff}_\lambda &= 5.347(15) \,, \\ \text{Cond\_eff}_{k-\lambda} &= 5.714(15) \,, \end{aligned}$$

to indicate

$$\text{Cond\_eff}_{k-\lambda} > \text{Cond\_eff}_\lambda \,. \tag{14.18}$$

However, when $m = 55 - 59$, we can see

$$\text{Cond\_eff}_{k-\lambda} = \text{Cond\_eff}_\lambda \,. \tag{14.19}$$

Both (14.18) and (14.19) are consistent with Corollary 12.2.

Next, let us choose the good λ. Take the solution by MFS in Tables 17 and 18 at $N = 71$ and $M = 50$, we can see

$$\max_{ij} |c_{ij}| = O(10^9) \,, \quad \sigma_{max} = 1.34 \,, \quad \sigma_{min} = 4.87(-26) \,.$$

Hence, we have

$$\sqrt{\sigma_{min}\sigma_{max}} = \sqrt{1.34 \times 4.87(-26)} = 2.55(-13) \,. \tag{14.20}$$

Based on Lemmas 13.1 and 13.2, we choose λ satisfying (13.28), i.e.,

$$4.87(-26) < \lambda < 2.55(-13)\,.$$

In computation, we use

$$\lambda = 1.0 \times 10^{-p}\,, \quad p = -26, -25, \cdots, -12\,,$$

and obtain the solutions by the MFS using the Tikhonov regularization. The results are listed in Table 22. Based on Table 22, we draw the curves in Figures 10–15. The curves in Figures 10–12 look not really as the L-shaped. If the errors

$$\frac{\|\mathbf{A}\mathbf{x}_k - \mathbf{b}\|}{\|\mathbf{A}\mathbf{x}_0 - \mathbf{b}\|} = O(10^2)\,, \tag{14.21}$$

is permitted, the optimal λ can be found by the corners of the curves. We can see

$$\lambda_{opt} \approx 1.0 \times 10^{-15}\,, \tag{14.22}$$

is for three curves in Figures 10–12. Interestingly, the optimal λ_{opt} is closer to $\sqrt{\sigma_{min}\sigma_{max}}$, but not to σ_{min}.

From (13.32), we have where $\mathbf{A} = \dfrac{\|\mathbf{A}\mathbf{x}_k - \mathbf{b}\|}{\|\mathbf{A}\mathbf{x}_0 - \mathbf{b}\|}$, $C = \dfrac{\text{Cond}_\lambda}{\text{Cond}}$. Hence, the good initial value of λ can be found by interpolation from the curve in Figure 14, under a given w. Then the exact solution $z = \lambda^2$ can be obtained by the Newton teration method.

To examine Theorem 13.3, we can see from Table 22,

$$\|\varepsilon\|_B \times \text{Cond\_eff}_\lambda = \|\mathbf{A}\mathbf{x}_k - \mathbf{b}\| \times \text{Cond\_eff}_\lambda = O(1)\,,$$

in $\lambda \in [1.0(-26), 1.0(-16)]$.

For the combination of Tikhonov regularization and TSVD, the computed results are listed in Tables 23–25. The behavior of decreasing of $\text{Cond}_{k-\lambda}$ is similar.

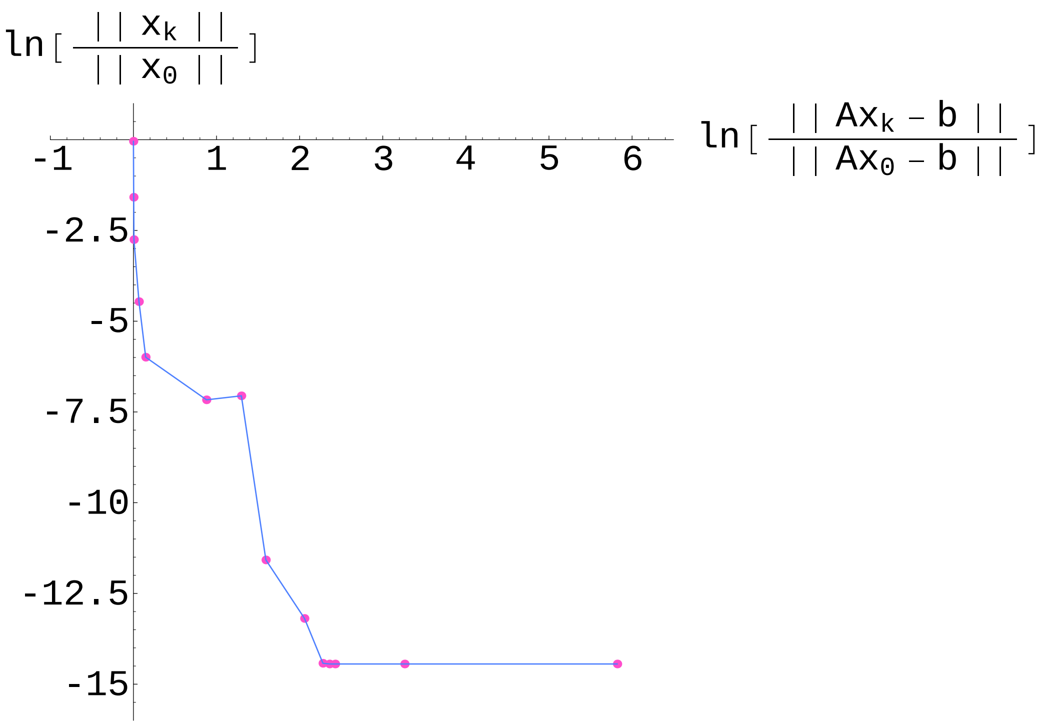

Figure 10: The curve of (x, y), where $x = \frac{\|\mathbf{A}\mathbf{x}_k - \mathbf{b}\|}{\|\mathbf{A}\mathbf{x}_0 - \mathbf{b}\|} = A$, $y = \frac{\|\mathbf{x}_k\|}{\|\mathbf{x}_0\|} = B$.

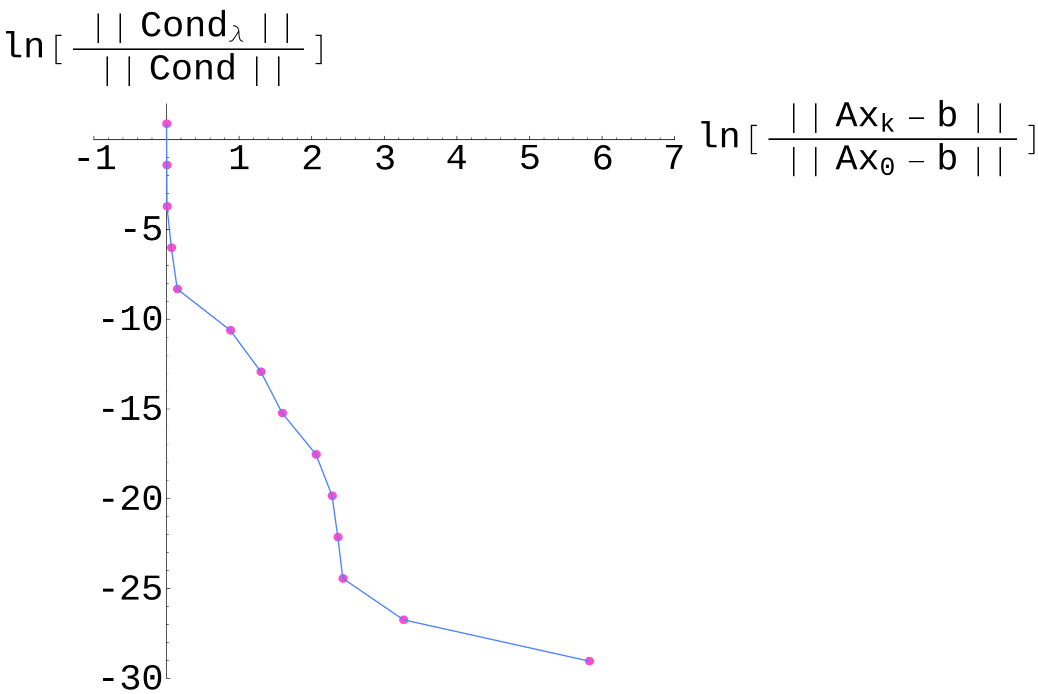

Figure 11: The curve of (x, y), where $x = \frac{\|\mathbf{Ax}_k - \mathbf{b}\|}{\|\mathbf{Ax}_0 - \mathbf{b}\|} = A$, $y = \frac{\text{Cond}_\lambda}{\text{Cond}} = C$.

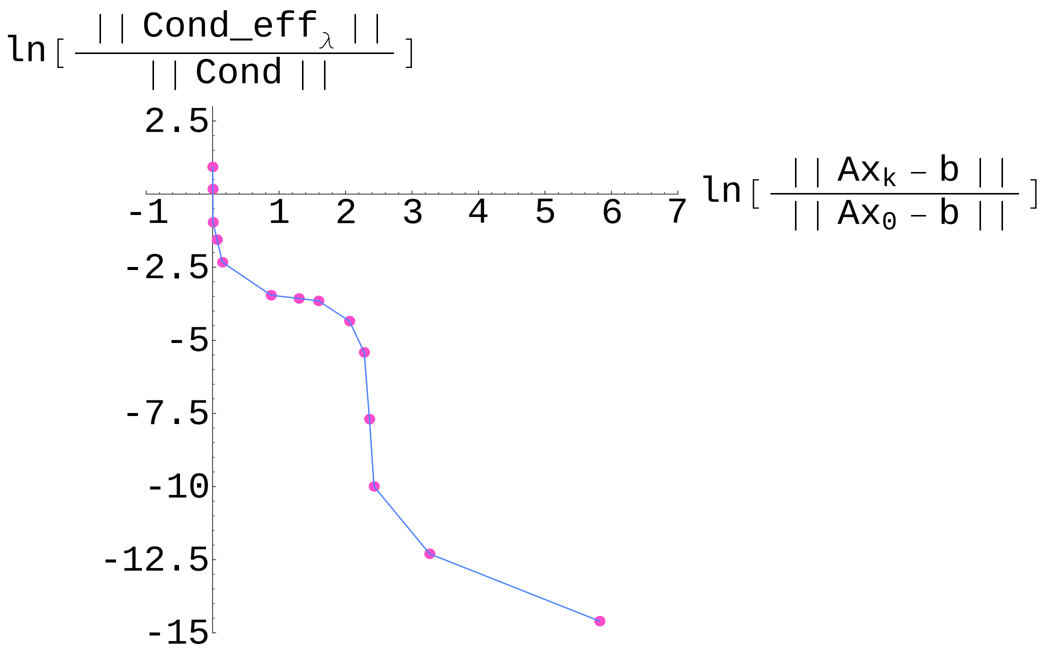

Figure 12: The curve of (x, y), where $x = \frac{\|\mathbf{Ax}_k - \mathbf{b}\|}{\|\mathbf{Ax}_0 - \mathbf{b}\|} = A$, $y = \frac{\text{Cond\_eff}_\lambda}{\text{Cond\_eff}} = D$.

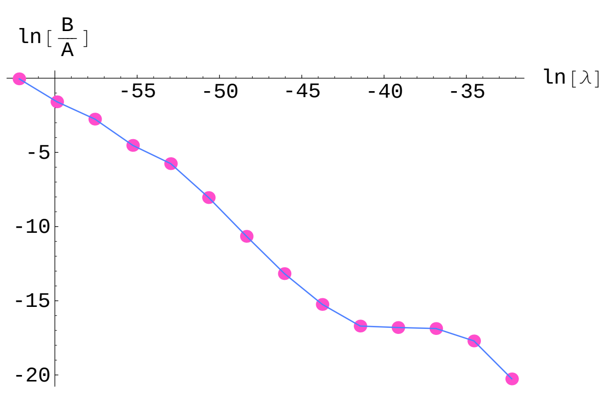

Figure 13: The curve of $(\lambda, \frac{B}{A})$, where $\frac{\|\mathbf{Ax}_k - \mathbf{b}\|}{\|\mathbf{Ax}_0 - \mathbf{b}\|} = A$, $\frac{\|\mathbf{x}_k\|}{\|\mathbf{x}_0\|} = B$.

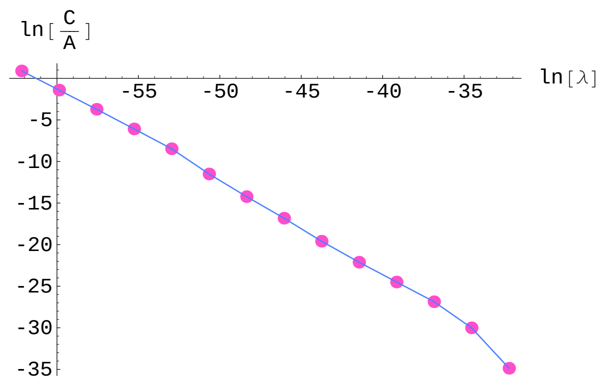

Figure 14: The curve of $(\lambda, \frac{C}{A})$, where $\frac{\|\mathbf{A}\mathbf{x}_k - \mathbf{b}\|}{\|\mathbf{A}\mathbf{x}_0 - \mathbf{b}\|} = A$, $\frac{\mathrm{Cond}_\lambda}{\mathrm{Cond}} = C$.

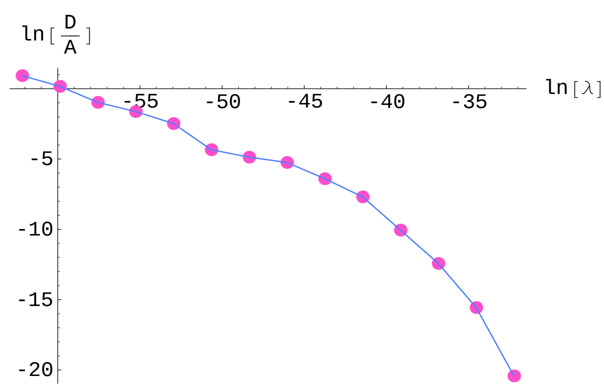

Figure 15: The curve of $(\lambda, \frac{D}{A})$, where $\frac{\|\mathbf{A}\mathbf{x}_k - \mathbf{b}\|}{\|\mathbf{A}\mathbf{x}_0 - \mathbf{b}\|} = A$, $\frac{\mathrm{Cond\_eff}_\lambda}{\mathrm{Cond\_eff}} = D$.

N	28	42	56	71
M	20	30	40	50
$\|\varepsilon\|_B$	0.272(-5)	0.122(-8)	0.673(-12)	0.171(-15)
β_n	0*	0.500(-9)	0*	-0.161(-16)
$\|\mathbf{x}\|$	0.143(4)	0.139(7)	0.267(7)	0.331(9)
$\|\mathbf{b}\|$	2.579	2.108	1.826	1.633
σ_{max}	1.331	1.331	1.331	1.340
σ_{min}	0.414(-10)	0.360(-15)	0.241(-20)	0.487(-25)
Cond	0.321(11)	0.370(16)	0.552(21)	0.275(26)
Cond_eff	0.435(8)	0.421(10)	0.283(15)	0.101(18)

Table 16: The error norms and condition numbers by the FSM for the smooth problem with $R = \sqrt{3}$.

c_1	-0.783733519934D+01	c_{36}	0.440576353157D+01
c_2	-0.102280662523D+02	c_{37}	0.286204204256D+01
c_3	-0.114862183885D+02	c_{38}	0.179970739343D+01
c_4	-0.113868075690D+02	c_{39}	0.108718870528D+01
c_5	-0.303835670518D+02	c_{40}	0.581870381328
c_6	-0.185833387272D+03	c_{41}	0.156274847163
c_7	-0.930561841471D+03	c_{42}	-0.238237469785
c_8	-0.243465860802D+04	c_{43}	-0.363056454233
c_9	0.188098296538D+05	c_{44}	0.662023593891
c_{10}	-0.318623453613D+05	c_{45}	0.391848896791D+01
c_{11}	-0.530340842574D+05	c_{46}	0.671226603159D+01
c_{12}	0.169405166582D+07	c_{47}	-0.205007141235D+01
c_{13}	-0.978449805029D+07	c_{48}	-0.265445267661D+02
c_{14}	0.317199524303D+08	c_{49}	-0.291545865018D+02
c_{15}	-0.722318212256D+08	c_{50}	0.279896773751D+02
c_{16}	0.124672634921D+09	c_{51}	0.540033820328D+02
c_{17}	-0.166107973002D+09	c_{52}	-0.528550589446D+01
c_{18}	0.172194102132D+09	c_{53}	-0.932247410555D+01
c_{19}	-0.140904305196D+09	c_{54}	0.190488081291D+02
c_{20}	0.916549264792D+08	c_{55}	-0.295808242306D+02
c_{21}	-0.457706539913D+08	c_{56}	-0.509291688926D+02
c_{22}	0.157186615239D+08	c_{57}	0.245846612813D+01
c_{23}	-0.286795653171D+07	c_{58}	0.326750884107D+02
c_{24}	0.249824339441D+05	c_{59}	0.137656212950D+02
c_{25}	0.664572253513D+05	c_{60}	-0.414779820326D+01
c_{26}	-0.230127171243D+04	c_{61}	-0.525927103999D+01
c_{27}	-0.872929951723D+04	c_{62}	-0.146341410578D+01
c_{28}	0.140353184972D+04	c_{63}	0.449567332910
c_{29}	0.624493285846D+03	c_{64}	0.608456587723
c_{30}	0.993995650393D+02	c_{65}	0.196241067936
c_{31}	0.158531391652D+02	c_{66}	-0.285318449009
c_{32}	0.112641742795D+02	c_{67}	-0.790803722062
c_{33}	0.109768382725D+02	c_{68}	-0.142716206459D+01
c_{34}	0.893321907229D+01	c_{69}	-0.233572663754D+01
c_{35}	0.648899897275D+01	c_{70}	-0.365951334406D+01
		c_{71}	-0.550537084733D+01

Table 17: All coefficients by the FSM for the smooth problem with $M = 50$ and $N = 71$, where the order of c_i.

i	σ_i	β_i	i	σ_i	β_i
1	1.34	5.59(-18)	36	1.01(-8)	1.82(-8)
2	7.98(-1)	6.43(-1)	37	6.39(-9)	-7.73(-11)
3	3.29(-1)	1.20(-16)	38	3.87(-9)	4.05(-9)
4	1.57(-1)	-7.18(-1)	39	2.43(-9)	-7.65(-11)
5	1.06(-1)	2.42(-15)	40	1.55(-9)	-2.02(-9)
6	3.69(-2)	-1.02	41	9.43(-10)	1.99(-10)
7	2.09(-2)	-2.49(-14)	42	6.02(-10)	-6.09(-10)
8	1.28(-2)	7.72(-1)	43	3.51(-10)	-1.32(-10)
9	6.76(-3)	7.83(-14)	44	2.31(-10)	-2.64(-10)
10	3.97(-3)	-2.73(-1)	45	1.22(-10)	3.74(-12)
11	2.41(-3)	-3.49(-13)	46	5.89(-11)	7.99(-11)
12	1.27(-3)	-1.20(-1)	47	4.06(-11)	-1.70(-11)
13	7.56(-4)	1.15(-13)	48	1.35(-11)	6.46(-12)
14	4.98(-4)	3.48(-2)	49	7.72(-12)	1.12(-11)
15	2.79(-4)	1.57(-12)	50	3.53(-12)	1.10(-12)
16	1.59(-4)	-1.02(-2)	51	8.76(-13)	6.22(-13)
17	1.03(-4)	1.32(-12)	52	6.65(-13)	-9.20(-13)
18	6.23(-5)	2.89(-3)	53	1.53(-13)	-2.88(-15)
19	3.57(-5)	3.96(-12)	54	4.05(-14)	-6.18(-14)
20	2.28(-5)	7.63(-4)	55	1.98(-14)	4.16(-15)
21	1.41(-5)	3.22(-12)	56	1.33(-15)	1.84(-15)
22	8.16(-6)	-1.54(-4)	57	2.09(-17)	-6.16(-16)
23	5.14(-6)	2.27(-11)	58	5.67(-18)	5.68(-16)
24	3.26(-6)	5.26(-5)	59	6.14(-20)	-1.30(-15)
25	1.91(-6)	-9.83(-12)	60	1.96(-19)	3.42(-16)
26	1.18(-6)	6.31(-6)	61	7.25(-20)	-3.64(-16)
27	7.61(-7)	-4.28(-12)	62	1.25(-20)	-5.01(-16)
28	4.55(-7)	-3.10(-6)	63	7.58(-21)	2.73(-16)
29	2.77(-7)	-4.62(-12)	64	7.06(-22)	5.31(-16)
30	1.79(-7)	1.38(-7)	65	1.97(-22)	-4.34(-17)
31	1.09(-7)	4.56(11)	66	8.48(-23)	-6.43(-17)
32	6.60(-8)	2.01(-7)	67	2.22(-23)	2.71(-17)
33	4.24(-8)	-8.91(-12)	68	1.17(-23)	2.43(-17)
34	2.64(-8)	1.73(-8)	69	1.02(-23)	6.10(-17)
35	1.59(-8)	-1.04(-11)	70	2.42(-24)	5.68(-17)
			71	4.87(-26)	-1.61(-17)

Table 18: All σ_i and β_i by the FSM for the smooth problem for $N = 71, M = 50$.

$k(\sigma_k)$	71	69	65	59	57	55
$\|\varepsilon\|_B$	0.171(-15)	0.181(-15)	0.204(-15)	0.949(-15)	0.171(-14)	0.259(-14)
$\|\mathbf{x}_0 - \mathbf{x}_k\|$	0	0.331(9)	0.331(9)	0.331(9)	0.331(9)	0.331(9)
$\frac{\|\mathbf{x}_0 - \mathbf{x}_k\|}{\|\mathbf{x}_0\|}$	0	0.999	0.999	1	1	1
$\|\mathbf{x}_k\|$	0.331(9)	0.653(7)	0.785(6)	0.213(4)	179	177
σ_{max}	1.34	1.34	1.34	1.34	1.34	1.34
σ_k	0.487(-25)	0.102(-22)	0.197(-21)	0.614(-18)	0.209(-16)	0.198(-13)
Cond_k	0.275(26)	0.131(24)	0.679(22)	0.218(19)	0.639(17)	0.678(14)
Cond_eff_k	0.101(18)	0.245(17)	0.105(17)	0.125(16)	0.434(15)	0.468(12)

$k(\sigma_k)$	51	50
$\|\varepsilon\|_B$	0.922(-12)	0.111(-11)
$\|\mathbf{x}_0 - \mathbf{x}_k\|$	0.331(9)	0.331(9)
$\frac{\|\mathbf{x}_0 - \mathbf{x}_k\|}{\|\mathbf{x}_0\|}$	1	1
$\|\mathbf{x}_k\|$	177	177
σ_{max}	1.34	1.34
σ_k	0.876(-12)	0.353(-11)
Cond_k	0.153(13)	0.379(12)
Cond_eff_k	0.105(11)	0.262(10)

Table 19: Using TSVD by the FSM for smooth problem with $N = 71, M = 50$ and σ_k.

c_1	-0.188985043984	c_{36}	0.306554057211
c_2	-0.121913576026	c_{37}	0.402060053648
c_3	-0.107976967495	c_{38}	0.448791719215
c_4	-0.645200911568D-02	c_{39}	0.311418758810
c_5	-0.856101661421D-01	c_{40}	-0.286366247736
c_6	0.442474264016	c_{41}	-0.177411828155D+01
c_7	-0.195934711995	c_{42}	-0.468402018845D+01
c_8	0.116319233845D+01	c_{43}	-0.928563167452D+02
c_9	0.252221452417D+01	c_{44}	-0.150803670403D+02
c_{10}	0.163665171457D+01	c_{45}	-0.198122904105D+02
c_{11}	-0.240800056431	c_{46}	-0.197732756634D+02
c_{12}	-0.185711101296D+01	c_{47}	-0.990489710581D+01
c_{13}	-0.278953182766D+01	c_{48}	0.110594041174D+02
c_{14}	-0.302822115517D+01	c_{49}	0.403851389265D+02
c_{15}	-0.268899665172D+01	c_{50}	0.652535195496D+02
c_{16}	-0.191825649060D+01	c_{51}	0.728295743656D+02
c_{17}	-0.870425096374	c_{52}	0.535795189844D+02
c_{18}	0.296712731793	c_{53}	0.118198429974D+02
c_{19}	0.142388548607D+01	c_{54}	-0.345012961799D+02
c_{20}	0.235208623505D+01	c_{55}	-0.665679646643D+02
c_{21}	0.292405057343D+01	c_{56}	-0.722448817337D+02
c_{22}	0.298601268299D+01	c_{57}	-0.539354028851D+02
c_{23}	0.239686041329D+01	c_{58}	-0.255566803039D+02
c_{24}	0.107746213195D+01	c_{59}	0.982189150179
c_{25}	-0.801664652992	c_{60}	0.161479055763D+02
c_{26}	-0.238036666041D+01	c_{61}	0.207720520049D+02
c_{27}	-0.200527111508D+01	c_{62}	0.177427268931D+02
c_{28}	-0.944131157334D-01	c_{63}	0.121433453427D+01
c_{29}	-0.456041962108	c_{64}	0.676651961951D+01
c_{30}	-0.237683652418D-01	c_{65}	0.302721251793D+01
c_{31}	0.874937994310D-02	c_{66}	0.881963425288
c_{32}	0.765220710541D-01	c_{67}	-0.909460685168D-01
c_{33}	0.111057638020	c_{68}	-0.418627296157
c_{34}	0.156460027447	c_{69}	-0.434889969980
c_{35}	0.220135616571	c_{70}	-0.357308900308
		c_{71}	-0.256498190200

Table 20: All coefficients by the FSM for the smooth problem for $N = 71, M = 50$ and σ_{57} with TSVD.

$m(\lambda=\sigma_m)$	71	69	65	59	57	55
$\|\varepsilon\|_B$	0.171(-15)	0.183(-15)	0.206(-15)	0.114(-14)	0.172(-14)	0.123(-13)
$\|\mathbf{x}_\lambda\|$	0.167(9)	0.374(7)	0.718(6)	0.109(4)	178	177
$\frac{\|\mathbf{x}_\lambda\|}{\|\mathbf{x}_0\|}$	0.504	0.113(-1)	0.217(-2)	0.331(-5)	0.536(-6)	0.534(-6)
σ_{max}	1.34	1.34	1.34	1.34	1.34	1.34
σ_m	0.489(-25)	0.102(-22)	0.197(-21)	0.614(-18)	0.209(-16)	0.198(-14)
Cond_λ	0.138(26)	0.655(23)	0.339(22)	0.109(19)	0.319(17)	0.339(14)
Cond_eff_λ	0.100(18)	0.213(17)	0.576(16)	0.121(16)	0.219(15)	0.234(12)

$m(\lambda=\sigma_m)$	51	50
$\|\varepsilon\|_B$	0.683(-12)	0.234(-11)
$\|\mathbf{x}_\lambda\|$	177	177
$\frac{\|\mathbf{x}_\lambda\|}{\|\mathbf{x}_0\|}$	0.534(-6)	0.534(-6)
σ_{max}	1.34	1.34
σ_m	0.876(-12)	0.353(-11)
Cond_λ	0.765(12)	0.189(12)
Cond_eff_λ	0.527(10)	0.131(9)

Table 21: Using Tikhonov regularization by the FSM for smooth problem with $N = 71, M = 50$ and $\lambda = \sigma_m$.

λ	1(-26)	1(-25)	1(-24)	1(-23)	1(-22)	1(-21)	1(-20)
$\|\varepsilon\|_B$	1.706(-16)	1.711(-16)	1.716(-16)	1.824(-16)	1.976(-16)	4.111(-16)	6.254(-16)
$\|\mathbf{x}_\lambda\|$	3.179(8)	6.785(7)	2.113(7)	3.825(6)	8.281(5)	2.559(5)	2.856(4)
Cond_λ	6.700(25)	6.700(24)	6.700(23)	6.700(22)	6.700(21)	6.700(20)	6.700(19)
Cond_eff_λ	2.568(17)	1.203(17)	3.864(16)	2.135(16)	9.860(15)	3.190(15)	2.859(15)
$\|\varepsilon\|_B \times \text{Cond\_eff}_\lambda$	43.82	20.59	6.629	3.894	1.949	1.312	1.788
$\frac{\|\mathbf{A}\mathbf{x}_k-\mathbf{b}\|}{\|\mathbf{A}\mathbf{x}_0-\mathbf{b}\|}$	1.000	1.003	1.006	1.069	1.159	2.409	3.665
$\frac{\|\mathbf{x}_k\|}{\|\mathbf{x}_0\|}$	0.959	0.205	6.380(-2)	1.155(-2)	2.499(-3)	7.726(-4)	8.624(-4)
$\frac{\text{Cond}_\lambda}{\text{Cond}}$	2.435	0.243	2.435(-2)	2.435(-3)	2.435(-4)	2.435(-5)	2.435(-6)
$\frac{\text{Cond\_eff}_\lambda}{\text{Cond\_eff}}$	2.537	1.189	3.816(-1)	2.108(-1)	9.739(-2)	3.151(-2)	2.823(-2)
$\frac{B}{A}$	0.959	0.204	6.345(-2)	1.080(-2)	2.158(-3)	3.207(-4)	2.353(-5)
$\frac{C}{A}$	2.435	0.243	2.421(-2)	2.277(-3)	2.101(-4)	1.010(-5)	6.643(-7)
$\frac{D}{A}$	2.537	1.185	3.795(-1)	1.972(-1)	8.406(-2)	1.308(-2)	7.703(-3)

λ	1(-19)	1(-18)	1(-17)	1(-16)	1(-15)	1(-14)	1(-13)	1(-12)
$\|\varepsilon\|_B$	8.398(-16)	1.337(-15)	1.669(-15)	1.808(-15)	1.935(-15)	4.463(-15)	5.766(-14)	7.601(-13)
$\|\mathbf{x}_\lambda\|$	3106	619.8	180.2	176.8	176.8	176.8	176.8	176.8
Cond_λ	6.700(18)	6.700(17)	6.700(16)	6.700(15)	6.700(14)	6.700(13)	6.700(12)	1.027(13)
Cond_eff_λ	2.629(15)	1.317(15)	4.530(14)	4.618(13)	4.619(12)	4.619(11)	4.619(10)	4.619(9)
$\|\varepsilon\|_B \times \text{Cond\_eff}_\lambda$	2.208	1.761	0.756	8.349(-2)	8.937(-3)	2.061(-3)	2.663(-3)	3.511(-3)
$\frac{\|\mathbf{A}\mathbf{x}_k-\mathbf{b}\|}{\|\mathbf{A}\mathbf{x}_0-\mathbf{b}\|}$	4.922	7.835	9.779	10.59	11.34	26.16	337.9	4455
$\frac{\|\mathbf{x}_k\|}{\|\mathbf{x}_0\|}$	9.377(-6)	1.871(-6)	5.441(-7)	5.337(-7)	5.337(-7)	5.337(-7)	5.337(-7)	5.337(-7)
$\frac{\text{Cond}_\lambda}{\text{Cond}}$	2.435(-7)	2.435(-8)	2.435(-9)	2.435(-10)	2.435(-11)	2.435(-12)	2.435(-13)	3.731(-13)
$\frac{\text{Cond\_eff}_\lambda}{\text{Cond\_eff}}$	2.596(-2)	1.301(-2)	4.474(-3)	4.562(-4)	4.562(-5)	4.562(-6)	4.562(-7)	4.562(-8)
$\frac{B}{A}$	1.905(-6)	2.388(-7)	5.564(-8)	5.037(-8)	4.706(-8)	2.041(-8)	1.579(-9)	1.198(-11)
$\frac{C}{A}$	4.946(-8)	3.108(-9)	2.489(-10)	2.298(-11)	2.147(-12)	9.309(-14)	7.205(-16)	8.376(-17)
$\frac{D}{A}$	5.275(-3)	1.661(-3)	4.575(-4)	4.305(-5)	4.022(-6)	1.744(-7)	1.349(-9)	1.024(-11)

Table 22: Using Tikhonov by the FSM for smooth problem with $N = 71, M = 50$ and $\lambda = \sigma_m$.

$m(\lambda=\sigma_m)$	65	64	60	59	57	55
$\Vert\varepsilon\Vert_B$	0.384(-15)	0.472(-15)	0.172(-14)	0.183(-14)	0.261(-14)	0.276(-14)
$\Vert\mathbf{x}_0-\mathbf{x}_{k-\lambda}\Vert$	0.609(9)	0.609(9)	0.609(9)	0.609(9)	0.609(9)	0.609(9)
$\frac{\Vert\mathbf{x}_0-\mathbf{x}_{k-\lambda}\Vert}{\Vert\mathbf{x}_0\Vert}$	1	1	1	1	1	1
$\Vert\mathbf{x}_{k-\lambda}\Vert$	0.714(6)	0.523(6)	0.482(4)	0.158(4)	179	178
σ_{max}	1.33	1.33	1.33	1.33	1.33	1.33
σ_λ	0.200(-21)	0.903(-21)	0.786(-19)	0.401(-18)	0.963(-17)	0.748(-15)
σ_k	0.200(-21)	0.200(-21)	0.200(-21)	0.200(-21)	0.200(-21)	0.200(-21)
$\text{Cond}_{k-\lambda}$	0.332(22)	0.737(21)	0.847(19)	0.166(19)	0.691(17)	0.889(15)
$\text{Cond\_eff}_{k-\lambda}$	0.571(16)	0.173(16)	0.216(16)	0.129(16)	0.473(15)	0.613(13)

Table 23: Using TSVD+Tikhonov by the FSM for smooth problem with $N=71, M=50$, $k(\sigma_k)=65$ and $\lambda=\sigma_m$.

$k(\sigma_k)$	65	59	57	55	54
$m(\lambda=\sigma_m)$	64	58	56	54	53
$\Vert\varepsilon\Vert_B$	0.472(-15)	0.205(-14)	0.275(-14)	0.509(-14)	0.184(-12)
$\Vert\mathbf{x}_0-\mathbf{x}_{k-\lambda}\Vert$	0.609(9)	0.609(9)	0.609(9)	0.609(9)	0.609(9)
$\frac{\Vert\mathbf{x}_0-\mathbf{x}_{k-\lambda}\Vert}{\Vert\mathbf{x}_0\Vert}$	1	1	1	1	1
$\Vert\mathbf{x}_{k-\lambda}\Vert$	0.523(6)	0.892(3)	178	178	178
σ_{max}	1.33	1.33	1.33	1.33	1.33
σ_λ	0.903(-21)	0.108(-17)	0.734(-15)	0.306(-13)	0.225(-12)
σ_k	0.200(-21)	0.401(-18)	0.963(-17)	0.749(-15)	0.306(-13)
$\text{Cond}_{k-\lambda}$	0.737(21)	0.617(18)	0.907(15)	0.218(14)	0.296(13)
$\text{Cond\_eff}_{k-\lambda}$	0.173(16)	0.849(15)	0.625(13)	0.149(12)	0.204(11)

Table 24: Using Tikhonov+TSVD by the FSM for smooth problem with $N=71, M=50$, $k(\sigma_k)$ and $\lambda=\sigma_m$.

$k(\sigma_k)$	55	55	55	55
$m(\lambda=\sigma_m)$	55	54	53	52
$\Vert\varepsilon\Vert_B$	0.310(-14)	0.509(-14)	0.184(-12)	0.578(-12)
$\Vert\mathbf{x}_0-\mathbf{x}_{k-\lambda}\Vert$	0.609(9)	0.609(9)	0.609(9)	0.609(9)
$\frac{\Vert\mathbf{x}_0-\mathbf{x}_{k-\lambda}\Vert}{\Vert\mathbf{x}_0\Vert}$	1	1	1	1
$\Vert\mathbf{x}_{k-\lambda}\Vert$	178	178	178	178
σ_{max}	1.33	1.33	1.33	1.33
σ_λ	0.749(-15)	0.306(-13)	0.225(-12)	0.516(-12)
σ_k	0.749(-15)	0.749(-15)	0.749(-15)	0.749(-15)
$\text{Cond}_{k-\lambda}$	0.889(15)	0.218(14)	0.296(13)	0.129(13)
$\text{Cond\_eff}_{k-\lambda}$	0.625(13)	0.149(12)	0.204(11)	0.889(10)

Table 25: Using Tikhonov+TSVD by the FSM for smooth problem with $N=71, M=50$, $k(\sigma_k)$ and $\lambda=\sigma_m$.

15 Concluding Remarks

1. For Laplace's equation on disk domains, the eigenvalues of the stiffness matrix from MFS are derived in Sections 4 – 5 for the Dirichlet, the Neumann and the Robin boundary conditions. The results of eigenvalues for the Dirichlet conditions are consistent with Christransen [12, 13], while those of the Neumann and the Robin boundary conditions are new. The traditional condition number for all the three kinds of boundary conditions grows exponentially.

2. For the mixed type of boundary conditions of Laplace's equation on the bounded and simply-connected domains, the bounds of Cond by MFS are derived in Section 6. The same exponential rates are obtained. The new stability in this thesis may enhance the MFS.

3. The effective condition number for LSM with rank deficient is proposed in this thesis. The effective condition number is much smaller than the traditional condition number for Motz's problem, see Section 7. Moreover, the SVD is used to solve the discrete equations of MFS, to deal with singular matrices of the Neumann, the Dirichlet and the Robin when $R = 1$.

4. Numerical experiments are carried out for the two kinds of cases: (1) MFS for Motz's problem by adding singular functions, (2) MFS for Motz's problem by local refinements of collocation points. The effective condition number is advantageous over Cond for better measuring the effects of rounding errors. However, there occurs more a severe subtraction cancellation in the final harmonic solutions. A balance between the accuracy and the ill-conditioning must be taken for practical applications. For Motz's problem, the MFS adding particular solutions is more advantageous in both errors and stability.

5. MFS for the smooth problem, the solution is more accurate, but the ill-conditioning is worse and the norm $\|x\|$ is also large. By TSVD and TR the condition number and $\|x\|$ are used to explore the improvements.

6.From numerical experiments, the traditional condition number decreases exponentially by TSVD and TR, but the effective condition number dose not apparent decreases.

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