MATHEMATICAL METHODS
IN ENGINEERING

An Introduction to the Mathematical Treatment
of Engineering Problems

BY
THEODORE v. KÁRMÁN
Director of the Guggenheim Aeronautics Laboratory
California Institute of Technology

AND
MAURICE A. BIOT
Assistant Professor of Mechanics, Columbia University;
Honorary Professor, University of Louvain

FIRST EDITION
TENTH IMPRESSION

McGRAW-HILL BOOK COMPANY, INC.
NEW YORK AND LONDON
1940
Let us denote the cross section of the column by \( A \) and the specific weight of its material by \( \gamma \). In the buckled position (Fig. 12.3) the weight of the portion of the beam between the top and the cross section at \( x \) is in equilibrium with the resultant of the axial stresses and the shear force acting at \( x \). Therefore, if the inclination of the axis of the column to the vertical is \( \theta \), the shear force \( S \) is equal to \( S = \gamma A x \theta \). The bending moment is 
\[
M = -EI \frac{d\theta}{dx}
\]
where \( EI \) is the flexural rigidity of the column. Therefore, the shear force 
\[
S = \frac{dM}{dx} = -EI \frac{d^2\theta}{dx^2}
\]
Obviously we obtain the differential equation for \( \theta \):

\[
EI \frac{d^2\theta}{dx^2} = -\gamma A x \theta
\]
or

\[
\frac{d^2\theta}{dx^2} + \frac{\gamma A}{EI} x \theta = 0
\]  
(12.10)

If we put \( x \sqrt{\gamma A/EI} = \xi \), the differential equation (12.10) becomes

\[
\frac{d^2\theta}{d\xi^2} + \xi \theta = 0
\]  
(12.11)

This equation is identical with Eq. (7.14), Chapter II, and its general solution is [Eq. (7.15), Chapter II]

\[
\theta = \xi^m Z_{m/3}(\xi^{1/3})
\]  
(12.12)

where the symbol \( Z_{m/3} \) means the general solution of Bessel's differential equation of one-third order. The general solution (12.12) is a linear combination of two particular solutions; one is of the form [(Eq. 7.17), Chapter II]:

\[
\theta_1 = \xi^{m/3} (a_0 + a_1 \xi^3 + a_2 \xi^6 + \cdots)
\]  
(12.13)
and the other of the form [Eq. (7.18), Chapter II]:

\[
\theta_2 = b_0 + b_1 \xi^3 + b_2 \xi^6 + \cdots
\]  
(12.14)
Chap. VII

We have the boundary conditions $d\theta/dx = d\theta/d\xi = 0$ for $\xi = 0$ since the bending moment is zero at the top cross section and $\theta = 0$ for $x = l$ or $\xi = l\sqrt{\gamma A/EI}$, if $l$ is the height of the column and we assume that the column is clamped normal to a horizontal base. It follows from the first condition that $a_0 = 0$; therefore [cf. Eq. (7.20), Chapter II], $\theta_1 = 0$, and we are left with

$$\theta = b_0 + b_1\xi^3 + b_2\xi^6 + \cdots$$  \hspace{1cm} (12.15)

where [cf. Eq. (7.19), Chapter II]

$$b_1 = -\frac{b_0}{2 \cdot 3}, \quad b_2 = -\frac{b_1}{5 \cdot 6}, \quad \cdots$$

and, therefore,

$$\theta_1 = b_0 \left(1 - \frac{\xi^3}{6} + \frac{\xi^6}{180} - \cdots\right)$$  \hspace{1cm} (12.16)

The second boundary condition is satisfied if the expression in the parentheses vanishes for $\xi = l\sqrt{\gamma A/EI}$. Therefore, we have to calculate the roots of the equation

$$1 - \frac{\xi^3}{6} + \frac{\xi^6}{180} - \cdots = 0$$  \hspace{1cm} (12.17)

The first approximation is $\xi^3 = 6$; or $\xi = \sqrt[3]{6} = 1.817$. In order to calculate the next approximations, we use Graeffe's method (Chapter V, section 8). Let us cut off (12.17) after the term with $\xi^3$ and substitute $\xi^3 = 1/2$. Then we have, after dividing (12.17) by $\xi^3$,

$$z^2 - \frac{z}{6} + \frac{1}{180} = 0$$  \hspace{1cm} (12.18)

Then multiplying Eq. (12.18) by $z^2 + \frac{z}{6} + \frac{1}{180}$, we obtain

$$z^4 - \frac{z^2}{36} - \frac{1}{90} + \frac{1}{180^2} = 0$$

Hence, the largest root $z$ (corresponding to the smallest root $\xi$) is approximately

$$z = \sqrt[4]{\frac{1}{90}}$$
or
\[ \xi = \sqrt{60} = 1.98 \]
(The exact value of the root is 1.986.) Hence, the critical length of the column is
\[ l_c = 1.98 \frac{3EI}{\gamma A} \]  
(12.19)
For a filament of circular cross section of radius \( r \) we have \( I/A = r^4/4 \), and, therefore,
\[ l_c = 1.98 \frac{1}{\sqrt{4}} \left( \frac{E}{\gamma} \right)^{1/4} \]  
(12.20)
The length \( L = E/\gamma \) is the length of a filament which by its own weight would produce a tensile stress equal to \( E \); obviously, \( L \) is a characteristic length of the material. Then it is seen that
\[ l_c \sim r^{3/4}L^{1/4} \]  
(12.21)
By repeated multiplication of the two series,
\[ g(\xi) = 1 - \frac{\xi^4}{6} + \frac{\xi^6}{180} - \cdots \]
\[ g(-\xi) = 1 + \frac{\xi^4}{6} + \frac{\xi^6}{180} + \cdots \]
we can obtain approximations to the roots \( \xi_1, \xi_2, \cdots \) of \( g(\xi) = 0 \) in ascending order. These roots would give the loads corresponding to higher modes of buckling.

13. Buckling of an Elastically Supported Beam.—The differential equation for the deflection of an elastically supported uniform beam was given in section 4:
\[ EI \frac{d^4w}{dx^4} + kw = p(x) \]  
(13.1)
To investigate the buckling of such a beam under action of an axial force \( P \), we replace the latter by an equivalent lateral load. We have seen that the moment of an axial force \( P \) is equal to \( Pw \); according to (3.3) the load corresponding to a moment distribution \( M(x) \) is equal to \( p(x) = -d^2M/dx^2 \); hence the
transverse load which would produce a moment \( M(x) = Pw \) is equal to \( p(x) = -P \frac{d^2w}{dx^2} \). Therefore, Eq. (13.1) becomes

\[
EI \frac{d^4w}{dx^4} + kw = -P \frac{d^2w}{dx^2}
\]

or

\[
\frac{d^4w}{dx^4} + \frac{P}{EI} \frac{d^2w}{dx^2} + \frac{k}{EI} w = 0
\]  

We can also deduce this equation by considering the equilibrium of the beam in slightly curved shape. Denoting the radius of curvature by \( R \) and assuming an axial force \( P \) acting on two cross sections a unit distance apart, we obtain a resultant force normal to the axis of the beam which is equal to \( P/R \) or, with the approximation used in the beam theory, to \( -P \frac{d^2w}{dx^2} \).

The characteristic equation of the differential Eq. (13.3) is

\[
\lambda^4 + \frac{P}{EI} \lambda^2 + \frac{k}{EI} = 0
\]

or

\[
\lambda^2 = -\frac{P}{2EI} \pm \sqrt{\left(\frac{P}{EI}\right)^2 - \frac{k}{EI}}
\]

It is seen that the two values of \( \lambda^2 \) are real and negative when \( P > 0 \) and

\[
P^2 > 4kEI
\]

In this case all solutions are trigonometric functions.

We are especially interested in the buckling of a beam of infinite length; in this case we can limit ourselves to periodic solutions because any other solution would yield infinite deflection either at \( x = \infty \) or at \( x = -\infty \). Writing the periodic solution in the form

\[
w = C \sin \frac{2\pi(x - a)}{l}
\]

where \( a \) is an arbitrary constant and \( \lambda = 2\pi/l \), we obtain from Eq. (13.4)

\[
16\pi^4 \frac{P}{EI} \frac{4\pi^2}{l^2} + \frac{k}{EI} = 0
\]
or
\[ P = \frac{4\pi^2 EI}{l^2} + \frac{kI^2}{4\pi^2} \]  
\hspace{1cm} (13.8)

It is seen that in the case of the infinite beam we do not obtain distinct critical loads, but a certain range for the load \( P \) which is capable of holding the beam in a deflected shape. This critical range extends from a smallest value \( P_{\text{min}} \) to \( P = \infty \).

We obtain \( P_{\text{min}} \) by differentiation of the expression (13.8)
\[ \frac{dP}{dl} = -\frac{8\pi^2}{l^3}EI + \frac{2kl}{4\pi^2} = 0 \]  
\hspace{1cm} (13.9)

The critical wave length, i.e., the wave length produced by the smallest axial load which causes buckling, is given by
\[ l_0 = 2\pi \sqrt[4]{\frac{EI}{k}} \]  
\hspace{1cm} (13.10)

Substituting (13.10) in (13.8), we obtain [see Eq. (13.6)]
\[ P_{\text{min}} = 2\sqrt{kEI} \]  
\hspace{1cm} (13.11)

For loads \( P < P_{\text{min}} \) the straight form is stable. For loads \( P > P_{\text{min}} \) we obtain two different wave lengths \( l_1 \) and \( l_2 \) for each value of the buckling load (Fig. 13.1). However, these configurations do not occur unless some additional restraint is introduced, as the beam buckles when \( P \) reaches the value \( P_{\text{min}} \).

14. Combined Axial and Lateral Load Acting on the Spar of an Airplane Wing.—Figure 14.1 represents schematically the spar of an airplane wing hinged at the fuselage at \( A(x = 0) \) and supported by a hinged strut at \( B(x = l) \). The horizontal component of the strut force exerts an axial compression upon the spar, which is subjected to an additional direct bending load \( p \). We denote the bending moment due to \( p \) by \( M_p \); then the differential equation for the deflection \( w \) becomes*
\[ EI \frac{d^2w}{dx^2} = -Pw - M_p \]

*The load \( p \) and the deflection \( w \) are here measured positive upward. The bending moment on the left is positive counterclockwise.