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（以退化核求解含夾雜拉普拉斯與赫姆茲方程之

系統性解法(2/2)）

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A systematic solution for Helmholtz problems with circular hole and/or inclusions by using BIEs in conjunction with degenerate kernels

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Abstract

In this project, we extend the systematic formulation to the exterior Helmholtz problems with circular boundaries. Earthquake analysis for the site response of alluvial valley or canyon subject to the incident SH-wave is the main concern. Not only the cavities but also inclusions are considered. Stress concentration factor of the cavity under the ground surface is studied. Besides, the surface amplitudes are examined for the inclusion problems. Image concept and technique of superposition are utilized for half-plane problems. Numerical examples are given to verify the validity of the present approach.

Keywords: Helmholtz equation, degenerate kernel, null-field integral equation, Fourier series, SH-wave

摘要：本計劃提出一套系統性之方法來求解含圓形邊界之赫姆茲方程的外域問題。在地震分析上，沖積山谷或河谷受到 SH 波入射之地表反應是主要的分析重點。所以，不僅是圓形孔洞的問題，夾雜的問題也是討論的重點之一。在解決半平面問題上，我們使用了疊加的技巧以及映象法的觀念，故圓形孔洞上的應力集中因子以及夾雜表面的振幅分析都可做一驗證。利用一些數值算例來測試我們所研發的程式並驗證半解析法的可行性。我們的方法有五個優點：良態的模式、免除奇異積分、無邊界層效應、指數收斂與免網格切割。

關鍵字：赫姆茲方程、退化核、零場積分方程、傅利葉級數、SH 波。

1. Introduction

One of the major concerns of engineering seismology is to understand and explain vibrational response of the soil excited by earthquakes. The problem of the scattering and diffraction of SH-waves by a two-dimensional arbitrary number and location of cavities and inclusions in full and half-planes is revisited in this chapter by using our unified formulation. In 1971, Trifunac [1] has solved the problem of a single semi-circular alluvial valley subject to SH-wave. Later, Pao and Mao [2] have published a book on the stress concentration in 1972. In 1973, Trifunac [3] has also derived the closed-form solution of a single semi-circular canyon subject to the SH-wave. The earliest reference to a closed-form solution of the scattering and diffraction of the incident SH-wave by an underground inclusion exists in an article concerning an underground circular tunnel by Lee and Trifunac [4]. In order to extend to arbitrary shape inclusion problems, Lee and Manoogian [5] have used the weighted residual method to revisit the problem of scattering and diffraction of SH-wave with respect to an underground cavity of arbitrary shape in a two-dimensional elastic half-plane. In the following years, they extended to the half-plane problems with a inclusion of arbitrary shape [6]. According to the literature review, it is observed that exact solutions for boundary value problems are only limited for simple cases, *e.g.* half-plane problem with a semi-circular canyon, a cavity under half-plane problem and an inclusion. Therefore, proposing a systematic approach for solving exterior Helmholtz problems with circular boundaries of various numbers, positions and radii is our goal in this project. Our approach can deal with a cavity problem as a limiting case of an inclusion problem with zero shear modulus.

In this project, the boundary integral equation method (BIEM) is utilized to solve the half-plane radiation and scattering problems with circular boundaries. Not only Fourier series for boundary densities but also the degenerate kernel for fundamental solutions in the present formulation is incorporated into the null-field integral equation. The key idea is that we can distribute the null-field point exactly on the real boundary by using the appropriate degenerate kernels in the real computation. All the improper boundary integrals are free of calculating the principal values (Cauchy and Hadamard) in place of series sum. For the hypersingular equation, vector decomposition for the radial and tangential gradients is carefully considered, especially in the nonfocal case. A scattering problem subject to the incident wave is decomposed into two parts, incident plane wave field and radiation field. The radiation boundary condition is the minus quantity of incident wave function for matching the boundary condition of total wave for cavity. Not only the stress concentration of the cavity is addressed, but also the surface displacements of alluvial valley and inclusion problems are calculated in this project.

2. Degenerate kernels of BIE formulation for the Helmholtz problem

The governing equation of the incident SH-wave problem is the Helmholtz equation as shown below:

$$(\nabla^2 + k^2)u(\mathrm{x}) = 0, \quad \mathrm{x} \in \Omega, \tag{1}$$

where ∇^2, k and Ω are the Laplacian operator, the wave number, and the domain of interest, respectively. Based on the dual boundary integral formulation of the domain point [7], we have

$$2\pi u(\mathrm{x}) = \int_B T(\mathrm{s},\mathrm{x}) u(\mathrm{s}) dB(\mathrm{s}) - \int_B U(\mathrm{s},\mathrm{x}) t(\mathrm{s}) dB(\mathrm{s}), \quad \mathrm{x} \in \Omega, \tag{2}$$

$$2\pi \frac{\partial u(\mathrm{x})}{\partial \mathrm{n}_\mathrm{x}} = \int_B M(\mathrm{s},\mathrm{x}) u(\mathrm{s}) dB(\mathrm{s}) - \int_B L(\mathrm{s},\mathrm{x}) t(\mathrm{s}) dB(\mathrm{s}), \quad \mathrm{x} \in \Omega, \tag{3}$$

where s and x are the source and field points, respectively, $t(\mathrm{s})$ is the directional derivative of $u(\mathrm{s})$ along the outer normal direction at s, and n_x is the outward normal vector at the field point x. By moving the field point x to the smooth boundary, we have

$$\pi u(\mathrm{x}) = C.P.V. \int_B T(\mathrm{s},\mathrm{x}) u(\mathrm{s}) dB(\mathrm{s}) - R.P.V. \int_B U(\mathrm{s},\mathrm{x}) t(\mathrm{s}) dB(\mathrm{s}), \quad \mathrm{x} \in B, \tag{4}$$

$$\pi \frac{\partial u(\mathrm{x})}{\partial \mathrm{n}_\mathrm{x}} = H.P.V. \int_B M(\mathrm{s},\mathrm{x}) u(\mathrm{s}) dB(\mathrm{s}) - C.P.V. \int_B L(\mathrm{s},\mathrm{x}) t(\mathrm{s}) dB(\mathrm{s}), \quad \mathrm{x} \in B, \tag{5}$$

where the $R.P.V.$, $C.P.V.$ and $H.P.V.$ denote the Cauchy principal value, Riemann principal value and Hadamard principal value, respectively. By collocating the field point x outside the domain (including boundary), the null-field integral equations yield

$$0 = \int_B T(\mathrm{s},\mathrm{x}) u(\mathrm{s}) dB(\mathrm{s}) - \int_B U(\mathrm{s},\mathrm{x}) t(\mathrm{s}) dB(\mathrm{s}), \quad \mathrm{x} \in D^c \cup B, \tag{6}$$

$$0 = \int_B M(\mathrm{s},\mathrm{x}) u(\mathrm{s}) dB(\mathrm{s}) - \int_B L(\mathrm{s},\mathrm{x}) t(\mathrm{s}) dB(\mathrm{s}), \quad \mathrm{x} \in D^c \cup B, \tag{7}$$

where D^c is the complementary domain. The field point x in Eqs. (6) and (7) can be located in the D^c or exactly on the real boundary if appropriate degenerate kernels are expressed. The $U(\mathrm{s,x})$, $T(\mathrm{s,x})$, $L(\mathrm{s,x})$ and $M(\mathrm{s,x})$ represent the four kernel functions which is expanded into the degenerate forms can be found in [8].

3. Image technique for solving half-plane scattering problems

3.1 Image concept for half-plane problems

For the half-plane problem with a circular cavity and/or inclusion as shown in Figures 1, we extend the problem into a full plane with the scatter by using image concept such that our formulation can be applied. By applying the concept of even function, the symmetry condition is utilized to satisfy the traction free ($t = 0$) condition on the ground surface. We merge the half-plane domain into the full-plane problem by adding with the reflection wave. To solve the problem, the superposition technique is employed by introducing two plane waves, one is incident and the other is reflective, instead of only one incident wave. After taking the free body of full-plane problem through the ground surface, we obtain the desired solution which satisfies the Helmholtz equation and all the boundary conditions in the half-plane domain.

3.2 Superposition technique of scattering problem into incident wave field and radiation problems

For the scattering problem subject to the incident wave, this problem can be decomposed into two parts. One is the incident wave field and another is the radiation field as shown in Figures 1. The relations between two parts are shown below

$$w_t^M = w^i + w^r + w^M \, , \tag{8}$$

$$t_t^M = t^i + t^r + t^M \, , \tag{9}$$

where the "t_t^M" denotes the total field of matrix including radiation and scattering. The superscripts "i" and "r" are the incident and reflection waves and the "t^M" denotes the radiation part of matrix and needs to be solved. To match the boundary condition for the cavity case, the total traction is defined as $t_t^M = 0$. For the inclusion case, we have the two constraints of the continuity of displacement and equilibrium of traction along the kth interface ($B_k, k = 1, \cdots, N$) as shown below:

$$w_t^M = w^I \text{ and } \mu^M t_t^M = -\mu^I t^I \text{ on } B_k \, , \tag{10}$$

The radiation parts of matrix (w^M and t^M) and inclusion (w^I and t^I) can be solved by employing our method.

4. Illustrative examples and discussions

In order to check the validity of the present formulation, the limiting case of incident SH-wave reduces to the static case of Honein *et al*. [9] is conducted. For the incident SH-wave problem, one cavity in the infinite plane subject to SH-wave is solved and is compared with the Pao and Mow's analytical solution [2]. We also revisit the same problems of Lee and Manoogian [6], Trifunac [1] and Tsaur *et al*. [11] for the alluvial problem. All the numerical results are given below by using ten terms of Fourier series ($L = 10$).

Case1: Two circular cavities lie on the y-axis

Figures 2(a) and 2(b) shows the geometry of the two circles which radii are $a_1 = 1$ and $a_2 = 2$. For the static case, the displacement field of the anti-plane deformation is defined as:

$$w^\infty = \frac{\tau^\infty y}{\mu} . \tag{12}$$

In the dynamic case with traction free condition on the circular boundaries, we assume an incident SH-wave with amplitude of linear function in the y direction as:

$$w^i = \frac{\tau^\infty y}{\mu} e^{ikx} . \tag{13}$$

When k approaches zero, the problem is reduced to a static case where Honei *et al*.'s solution [9] can be compared with for $D = 2,\ 0.1$ and 0.01. Figures 2(e) and 2(f) show the graph of the stress $\sigma_{\theta z}$ around the boundary of smaller circle for various distances, D, between the two circles. Our numerical results are compared well with the data of Honein *et al.*'s data [9] when k approaches zero ($k = 0.001$) by using ten Fourier terms ($L = 10$).

Case2: A circular cylinder cavity in an infinite plane

Consider a circular cylinder with a radius "a" as shown in Figure 4(a). An incident SH-wave is defined by

$$u_x = 0, \quad u_y = 0, \quad u_z = W_0 e^{ikx}. \tag{14}$$

Figures 3(b) and 3(c) show the graph of the $\sigma_{\theta z}$ along the circular boundary for various wave numbers $ka = 0.1,\ 1.0$ and 2.0. It is worth noting that our data agree well with the analytical solution of Pao and Mow's data [2].

Case 3: Half-plane problem with two inclusions subject to the SH-wave

In order to verify that the present approach can be extended to handle arbitrary number and various positions of circular inclusions, we consider the problem with two inclusions under the ground surface subject to SH-wave as shown in Figure 4. Figure 5 shows the surface amplitude of the two-inclusions problem with $\mu^I/\mu^M = 1/6, \rho^I/\rho^M = 2/3$ for four cases of η (0.1, 0.25, 0.75 and 1.25). For the limiting case, the two inclusions problem reduces to two cavities problem when we set $\mu^I/\mu^M = 10^{-8}$. Good agreements are obtained after comparing with the results of two-cavities problem of Jiang *et al.* [12] as shown in Figure 6.

5. Concluding remarks

The first attempt to employ degenerate kernel in BIEM for problems with circular boundaries subject to the SH-wave was achieved. Not only cavity but also inclusion problems were treated. We have proposed a BIEM formulation by using degenerate kernels, null-field integral equation and Fourier series in companion with adaptive observer systems and vector decomposition. This method is a semi-analytical approach for problems with circular boundaries since only truncation error in the Fourier series is involved. Good agreements are obtained after comparing with previous results. The stress concentration factor of cavity case and the surface motion of half-plane problem with inclusions were determined. Parameter study on the surface amplitudes was also addressed. The analysis of amplification and interference effects for valley and inclusions subject to SH-waves may explain the ground motion either observed or recorded during earthquake. The method shows great generality and versatility for the problems with multiple circular cavities and inclusions of arbitrary radii and positions.

6. Evaluation

This study of 2-years project has been fulfilled on the Laplace, Helmholtz problems with circular boundaries. A summarized table is shown in Table 1. The research people and published journals are also listed in the Table 1.

7. References

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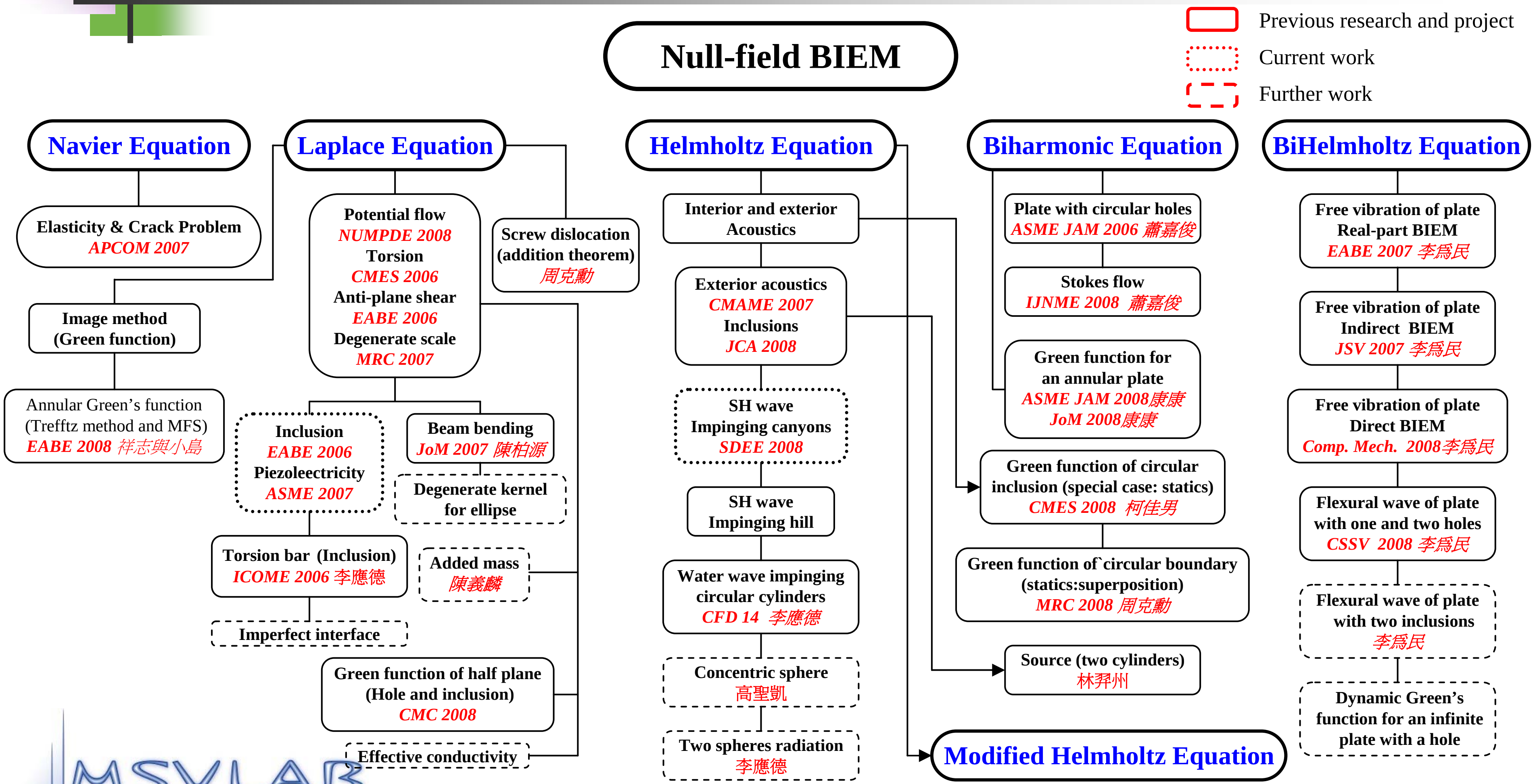
Table1 Research topics of NTOU / MSV LAB on null-field BIEMs (2003-2008)



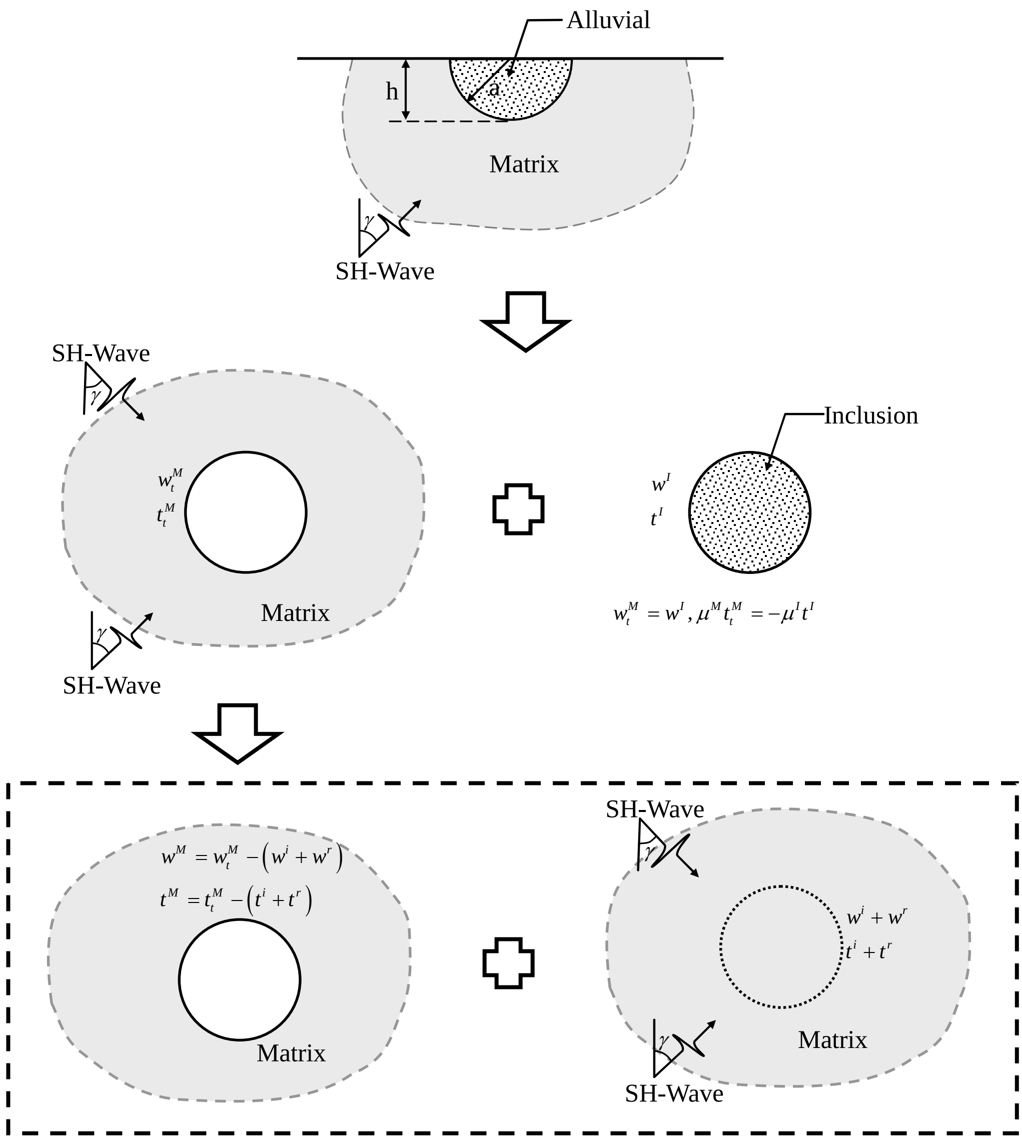
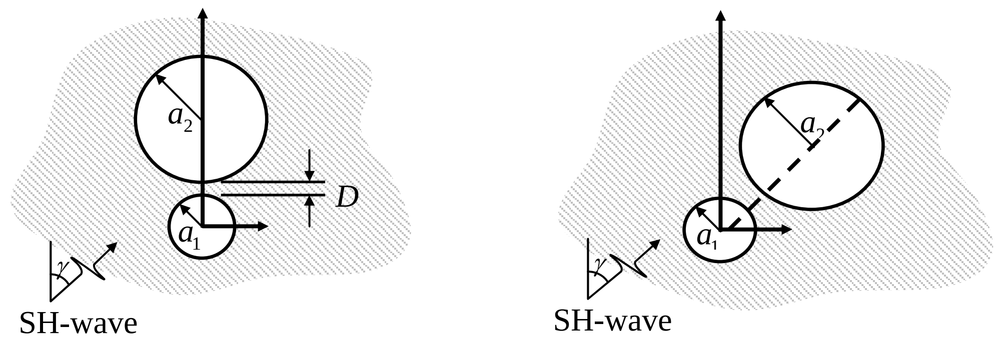
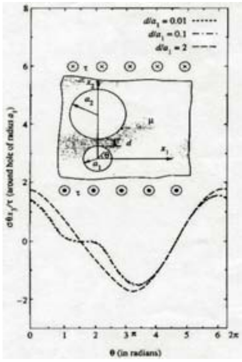


Figure 1 Image concept and the decomposition of superposition of a alluvial valley

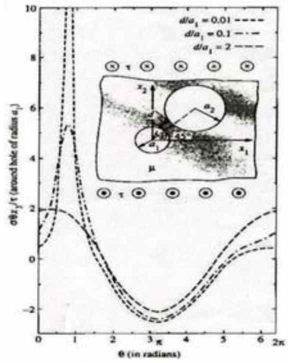


(a) Two cavities with centers on the $y-axis$

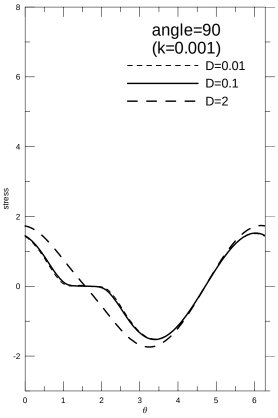
(b) Two cavities with centers on the $45^{\circ}-axis$



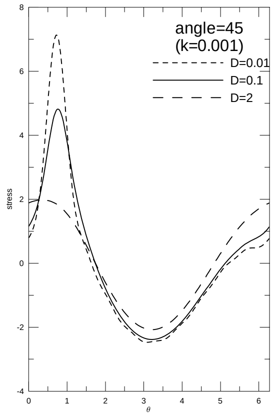
(c) Shear stress around the smaller cavity (Honein's result [9])



(d) Shear stress around the smaller cavity (Honein's result [9])

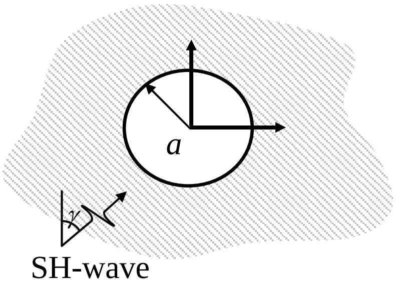


(e) Shear stress around the smaller cavity (Present method)

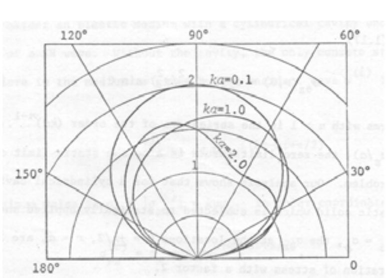


(f) Shear stress around the smaller cavity (Present method)

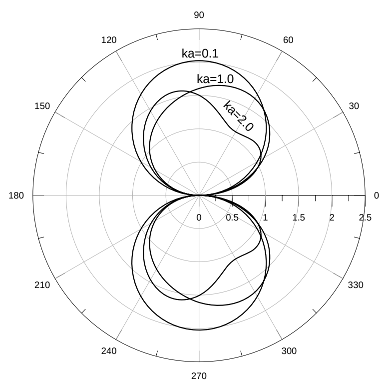
Figure 2 A full-plane problem with two cavities subject to the incident SH-wave.



(a) A full-plane problem with a cavity subject to SH-wave.

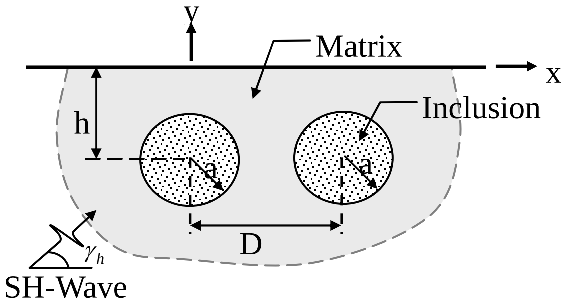


(b) Pao and Mow's result [2] (only half).



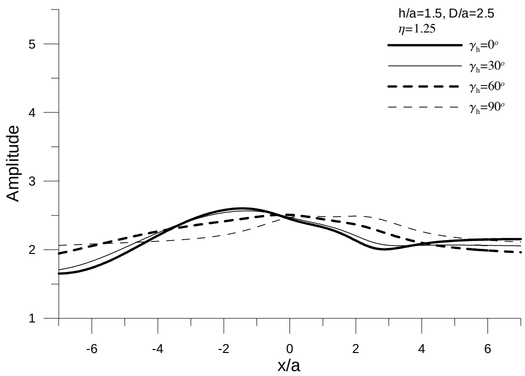
(c) Present method ($\gamma = 90^\circ$).

Figure 3 Shear stress ($\sigma_{\theta z}$) around the cavity of a full-plane problem subject to the horizontally incident SH wave.

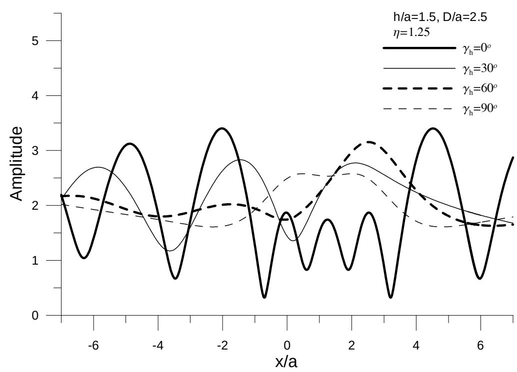


ρ^I : density of inclusion
ρ^M : density of matrix
μ^I : shear modulus of
inclusion
μ^M : shear modulus of

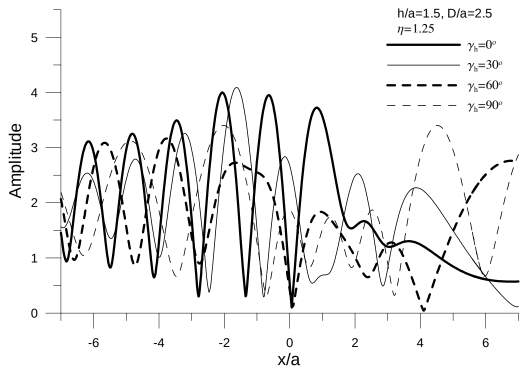
Figure 4 A half-plane problem with two circular inclusions subject to the SH-wave.



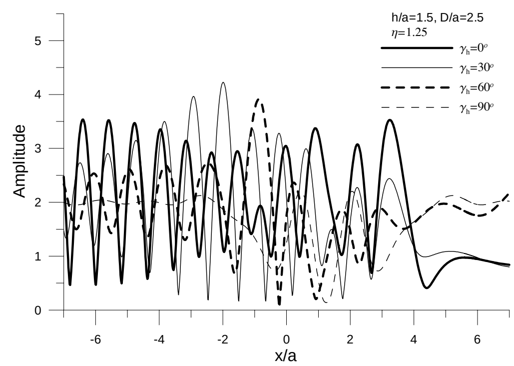
(a) $\eta = 0.1$



(b) $\eta = 0.25$

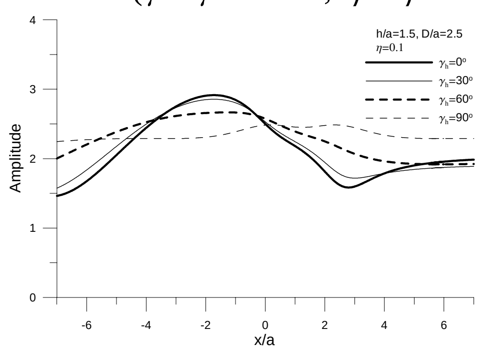


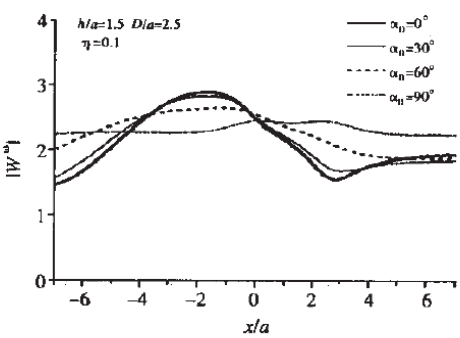
(c) $\eta = 0.75$



(d) $\eta = 1.25$

Figure 5 Surface amplitudes of two-inclusions problem
($\mu^I / \mu^M = 1/6$, $\rho^I / \rho^M = 2/3$, $h/a = 1.5$, $D/a = 2.5$, $L = 10$).

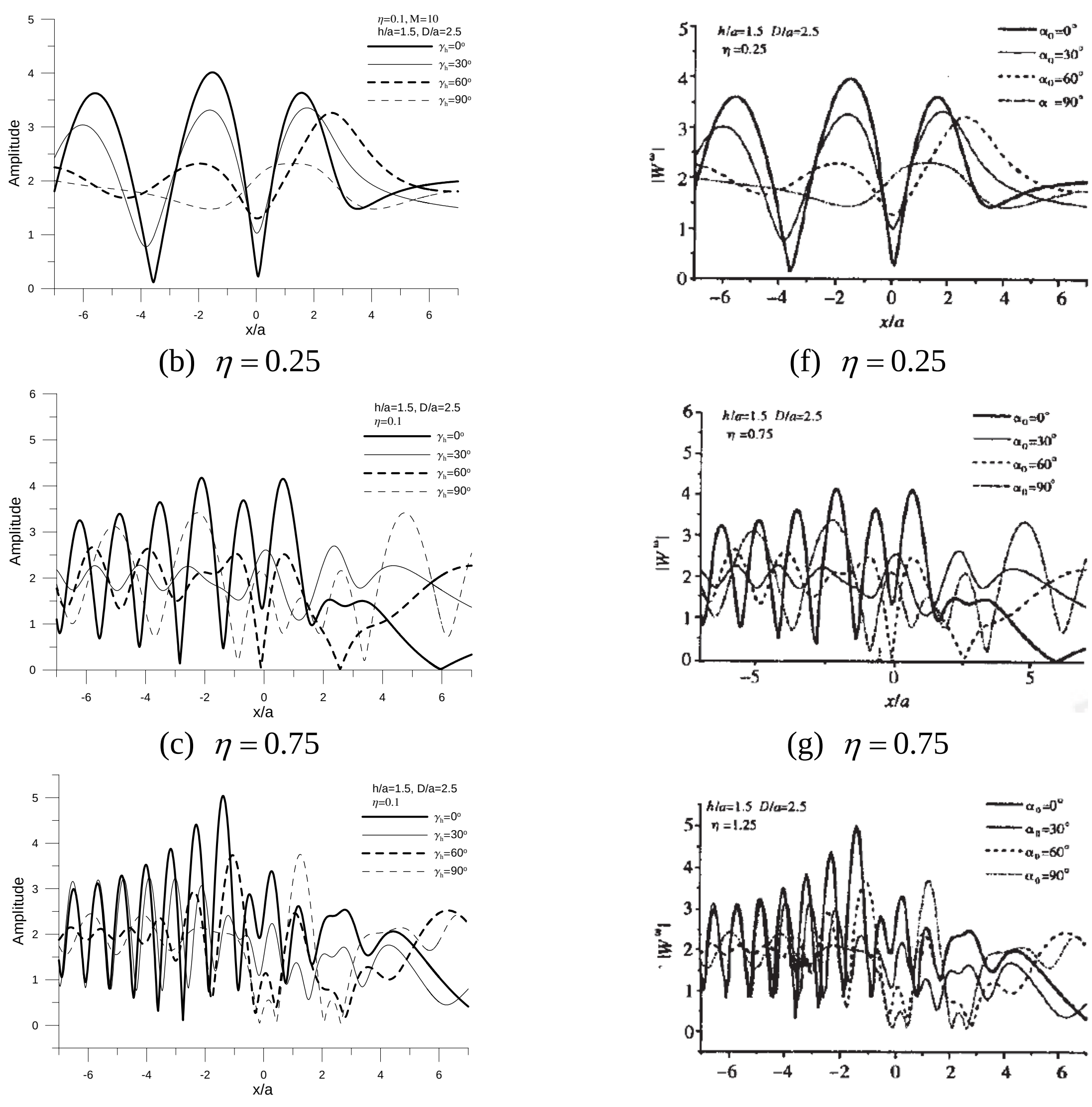




(a) $\eta = 0.1$ (e) $\eta = 0.1$

(b) $\eta = 0.25$ (f) $\eta = 0.25$

(c) $\eta = 0.75$ (g) $\eta = 0.75$



(d) $\eta = 1.25$ (h) $\eta = 1.25$

Figure 6 Limiting case of two-cavities problem

($\mu^I / \mu^M = 10^{-8}$, $\rho^I / \rho^M = 2/3$, $h/a = 1.5$, $D/a = 2.5$ and $L = 10$).

((a) ~ (d) Present method, (e) ~ (h) Jiang *et al.* result [12])