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Degenerate scale for a torsion bar problem using BEM

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Abstract

In this paper, a torsion bar problem is solved by using BEM. It is found that a degenerate scale problem occurs if the conventional BEM is used. In this case, the influence matrix is rank deficient and numerical results become unstable. Three regularization techniques, method of adding a rigid body mode, hypersingular formulation and CHEEF concept, are employed to deal with the rank-deficiency problem. The existence of degenerate scale is proved for the two-dimensional Laplace problem. The addition of a rigid body term, c, in the fundamental solution can shift the original degenerate scale to a new degenerate scale by a factor e^{-c}. A numerical example for circular bar with keyway under torsion, was demonstrated to show the failure of conventional BEM in case of the degenerate scale. After employing the three regularization techniques, the accuracy of the proposed approaches is achieved.

1. Introduction

In the BEM implementation, the rigid body motion or so called constant potential test is always employed to examine the singular matrices of strongly singular kernels and hypersingular kernels for the problems without degenerate boundaries. According to this concept, the diagonal terms of a singular influence matrix can be easily determined. Singular matrix occurs physically and mathematically in the sense that the nonunique solution for the singular matrix implies an arbitrary rigid body term for the interior Neumann problem. However, the influence matrix of the weakly singular kernel may be singular for the Dirichlet problem when the geometry is special. The nonunique solution is not physically realizable but results from the zero singular value of the influence matrix in the BEM. From the point of view of linear algebra, the problem also originates from the rank deficiency in the influence matrices. For example, the nonunique solution of a circle with a unit radius has been noted by Jaswon and Symm [4]. The special geometry which results in a nonunique solution for a potential problem is called "degenerate scale". Christiansen [2] termed it a critical value (C.V.) since it is mathematically realizable. In real implementation, we need to avoid the number one for the circular radius using the normalized scale. The numerical difficulties due to nonuniqueness of solutions have been solved by using the necessary and sufficient boundary integral equation (NSBIE) [3] and boundary contour method [6]. Chen *et al.* [1] studied the degenerate scale for the simply-connected and multiply-connected problems by using the degenerate kernels and circulants in a discrete system for circular and annular cases. Mathematically speaking, the singularity pattern distributed along a ring boundary resulting in a null-field solution introduces a degenerate scale. SVD technique has been used to detect the nonunique solution in case of degenerate scale [2].

In this study, we will propose three alternatives, method of adding a rigid body mode, hypersingular formulation and CHEEF technique, to overcome the nonunique solution in the numerical implementation. Method of adding a rigid body mode in the fundamental solution can shift the zero singular value in the conventional BEM. Instead of using the conventional BEM, the second equation in the dual BEM, i.e., hypersingular formulation, can avoid the zero singular value. By using the CHEEF technique, the addition of a constraint by collocating the point outside the domain can promote the rank of the singular matrix. A numerical example, torsion problem of circular bar with keyway, will be demonstrated to see the numerical instability for the degenerate scale problem. The treatment for the suppression of numerical instability will be done.

2. Dual boundary integral formulation and dual BEM for a torsion problem

The torsion problem of a bar with an arbitrary cross section in Fig.1 can be formulated by the Poisson equation as follows [5]:

$$\nabla^2 u^*(x_1, x_2) = -2, \qquad (x_1, x_2) \in D, \tag{1}$$

where u^* is the torsion (Prandtl) function, ∇^2 is the Laplacian operator and D is the domain. The boundary condition is

$$u^*(x_1, x_2) = 0, \qquad (x_1, x_2) \in B, \tag{2}$$

where B is the boundary. Since Eq.(1) contains the body source term which results in a domain integral by using the BEM, the problem can be reformulated to

$$\nabla^2 u(x_1, x_2) = 0, \qquad (x_1, x_2) \in D, \tag{3}$$

and the homogeneous boundary condition in Eq.(2) is changed to $u(x_1, x_2) = \frac{(x_1^2+x_2^2)}{2}$, where the torsion function u^* can be obtained from u by superimposing $\tilde{u}$, $u = u^* + \tilde{u}$ and $\tilde{u} = \frac{(x_1^2+x_2^2)}{2}$. This new model for the torsion problem using Eq.(3) is the Laplace equation subject to the Dirichlet data, $u(x_1, x_2) = \frac{(x_1^2+x_2^2)}{2}$, which is very easy to implement using the DBEM, e.g., the BEPO2D program can be used in this study. The torque, M_z, can then be determined by

$$M_z = \iint_D (x_1\tau_{23} - x_2\tau_{13})\, dx_1\, dx_2, \tag{4}$$

where τ_{23} and τ_{13} are the shearing stresses determined by $\tau_{23} = -\kappa G \frac{\partial u^*}{\partial x_1}$ and $\tau_{13} = \kappa G \frac{\partial u^*}{\partial x_2}$, G is the shear modulus and κ denotes the twist angle per unit length. By employing the Green's second identity and Eq.(1), the area integral in Eq.(4) can be transformed into a boundary integral and a domain integral as follows:

$$M_z = -\kappa G \oint_B \tilde{u} \frac{\partial u^*}{\partial n}\, dB - \kappa G \iint_D (x_1^2 + x_2^2)\, dx_1\, dx_2. \tag{5}$$

The induced area integral of the second term on the right hand side of the equal sign in Eq.(5) can be reformulated into a boundary integral again by using the Gauss theorem as follows:

$$-\kappa G \iint_D (x_1^2 + x_2^2)\, dx_1\, dx_2 = \frac{-\kappa G}{16} \oint_B \frac{\partial\{(x_1^2 + x_2^2)^2\}}{\partial n}\, dB. \tag{6}$$

The torsion problem can be simulated by using the mathematical model of the Laplace equation as shown in Eq.(3). Using the Green's identity, the first equation of the dual boundary regular integral equations for the domain point x can be derived as follows:

$$2\pi u(x) = \int_B T(s,x) u(s)\, dB(s) - \int_B U(s,x) \frac{\partial u(s)}{\partial n_s}\, dB(s), \tag{7}$$

where $U(s,x) = ln(r)$ and $T(s,x) = \frac{\partial U(s,x)}{\partial n_s}$, in which r is the distance between the field point x and the source point s, and n_s is the normal vector for the boundary point s. After taking the normal derivative of Eq.(7), the second equation of the dual boundary regular integral equations for the domain point x can be derived:

$$2\pi \frac{\partial u(x)}{\partial n_x} = \int_B M(s,x) u(s) dB(s) - \int_B L(s,x) \frac{\partial u(s)}{\partial n_s} dB(s), \tag{8}$$

where $L(s,x) = \frac{\partial U(s,x)}{\partial n_x}$ and $M(s,x) = \frac{\partial^2 U(s,x)}{\partial n_x \partial n_s}$, in which n_x is the normal vector for the field point x. Eqs. (7) and (8) are coined the dual boundary regular integral equations for the domain point x. By tracing the field point x to the boundary, the dual boundary singular integral equations for the boundary point x can be derived:

$$\pi u(x) = C.P.V. \int_B T(s,x)u(s)dB(s) - R.P.V. \int_B U(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \tag{9}$$

$$\pi \frac{\partial u(x)}{\partial n_x} = H.P.V. \int_B M(s,x)u(s)dB(s) - C.P.V. \int_B L(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \tag{10}$$

where *R.P.V.*, *C.P.V.* and *H.P.V.* denote the Riemann principal value, Cauchy principal value and Hadamard principal value, respectively. After discretizing the boundary into $2N$ boundary elements, Eqs. (9) and (10) reduce to [1]

$$[U]_{2N\times 2N}\{t\}_{2N\times 1} = [T]_{2N\times 2N}\{u\}_{2N\times 1}, \tag{11}$$

$$[L]_{2N\times 2N}\{t\}_{2N\times 1} = [M]_{2N\times 2N}\{u\}_{2N\times 1}, \tag{12}$$

where $\{u\}$ and $\{t\}$ are the boundary data for the primary and the secondary boundary variables, respectively. To determine the torsion rigidity using Eq.(5), the following boundary integral can be integrated numerically as follows:

$$\oint_B \tilde{u}\frac{\partial u^*}{\partial n}dB = \oint_B \tilde{u}\frac{\partial u}{\partial n}dB - \oint_B \tilde{u}\frac{\partial \tilde{u}}{\partial n}dB = \sum_{j=1}^{2N} \tilde{u}_j[(\frac{\partial u}{\partial n})_j - (\frac{\partial \tilde{u}}{\partial n})_j]\, l_j, \tag{13}$$

where $(\frac{\partial u}{\partial n})_j$ is the normal derivative of u for the j^{th} boundary element, l_j is the length of the j^{th} boundary element and another boundary integral in Eq.(6) can be discretized as follows:

$$\oint_B \frac{\partial\{(x_1^2 + x_2^2)^2\}}{\partial n}dB = 4\sum_{j=1}^{2N}(\frac{\partial \tilde{u}^2}{\partial n})_j\, l_j. \tag{14}$$

3. Proof of the existence theorem for the degenerate scale

Theorem:
For any two-dimensional Laplace problem with a simply-connected domain, there exists a degenerate scale when we solve the problem by using the boundary integral formulation or BEM.
Proof:
For two-dimensional potential problems, there exists a unique solution for $\psi_1(s)$ satisfying

$$u(x) = \int_B U(s,x)\psi_1(s)\, dB(s), \quad x \in B, \tag{15}$$

where B is the normal boundary with the enclosing domain D. For simplicity, we can assume a constant potential field since it is a "simple solution" for the Laplace equation. Eq.(15) reduces to

$$1 = \int_B U(s,x)\,\psi_1(s)\, dB(s), \quad x \in B. \tag{16}$$

When the degenerate scale B_d occurs, the nonunique solution of Eq.(16) implies that

$$0 = \int_{B_d} U(s,x)\,\psi_1(s)\, dB(s), \quad x \in B_d, \tag{17}$$

has a nontrivial solution for $\psi_1(s)$, where B_d is the boundary of degenerate scale using the fundamental solution $U(s,x) = \ln(r)$. By expressing the boundary contour in terms of $f(x_1, x_2) = 0$ as shown in

Fig.2, we have a new closed boundary curve, $f(\frac{x_1}{d}, \frac{x_2}{d}) = 0$. By mapping the nondegenerate (normal) boundary to the degenerate boundary, we have $(x_1, x_2) \Rightarrow (x_1\, d, x_2\, d) = (x_1, x_2)\, d$, $dB(s) \Rightarrow dB(s\, d) = dB(s)\, d$, $U(s, x) \Rightarrow U(s\, d, x\, d) = U(s, x) + \ln(d)$, $\psi_1(s) \Rightarrow \bar{\psi_1}(s\, d) = \psi_1(s)$. According to mapping properties, the homogeneous Eq.(17) yields

$$0 = \int_{B_d} U(s\, d, x\, d)\ \bar{\psi_1}(s\, d)\ dB(s\, d). \tag{18}$$

In order to have a nontrivial solution for Eq.(18), we have

$$d + d\ \ln(d)\ \Gamma = 0, \tag{19}$$

after using Eq.(16) and defining

$$\Gamma = \int_B \psi_1(s) dB(s). \tag{20}$$

According to Eq.(19), the degenerate scale occurs when the expansion ratio, d, satisfies $d = e^{-\frac{1}{\Gamma}}$. The degenerate scale can be determined using only one normal scale. □

4. Three regularization techniques to deal with a degenerate scale problem

4.1. Method of adding a rigid body mode

Since the $[U]$ matrix is singular in case of the degenerate scale, the modified fundamental solution can be added by a rigid body term c,

$$U^*(s, x) = U(s, x) + c. \tag{21}$$

The influence matrix $[U]$ is modified to $[U^*]$, where the component form for the element is

$$U^*_{ij} = U_{ij} + c\, l_j.\ \ (i,\ j = 1, \cdots 2N) \tag{22}$$

The zero singular value in $[U]$ moves to a nonzero value for $[U^*]$. To demonstrate the effectiveness, the minimum singular value using the modified fundamental solution will be shown in the numerical example.

4.2. Hypersingular formulation

Instead of using the Eq.(11) in the conventional BEM, Eq.(12) in the dual BEM is used. To demonstrate the idea, the singular value for the $[L]$ matrix will be shown to be nonzero no matter what the expansion ratio is in the following numerical example.

4.3. CHEEF method

Since the $[U]$ matrix is singular, the rank is deficient. In order to promote the rank, the independent constraint is required. To resort to the null field equation by collocating the point outside the domain, we have

$$\{w^U\}\ \{t\} = \{w^T\}\ \{u\}, \tag{23}$$

where $\{w^U\}$ and $\{w^T\}$ are the influence row vectors by collocating the exterior point in the null-field equation. By combining Eq.(11) with Eq.(23), we have

$$\left[\begin{array}{c} [U]_{2N\times 2N} \\ \{w^U\}_{1\times 2N} \end{array} \right] \{\ t\ \}_{2N\times 1} = \left[\begin{array}{c} [T]_{2N\times 2N} \\ \{w^T\}_{1\times 2N} \end{array} \right] \{\ u\ \}_{2N\times 1}. \tag{24}$$

According to the Eq.(24), we can obtain the reasonable solution by using either the least squares method or the SVD technique.

5. A numerical example

For the circular bar with keyway under torsion, the analytical solution for the conjugate warping function is [5]

$$u(x_1, x_2) = a x_1 (1 - \frac{b^2}{x_1^2 + x_2^2} + \frac{1}{2} b^2),\ \ (x_1, x_2) \in D, \tag{25}$$

The torsion rigidity, T_r, is

$$T_r = 2Ga^4 k_2, \tag{26}$$

where

$$k_2 = \frac{1}{24}(\sin 4\gamma + 8\sin 2\gamma + 12\gamma) - \frac{1}{2}(\frac{b}{a})^2(\sin 2\gamma + 2\gamma) + \frac{4}{3}(\frac{b}{a})^3(\sin\gamma) + \frac{1}{4}(\frac{b}{a})^4\gamma \tag{27}$$

in which $\cos\gamma = \frac{b}{2a}$. Table 1 shows the torsional rigidity by using different approaches. Also, the degenerate scale is found in Table 1 using Eq.(20). The conventional BEM (UT formulation) can not obtain the acceptable results for the case of degenerate scale as shown in Table 1. By using the conventional BEM, the zero singular value occurs in case of the degenerate scale. After adding the rigid body term in the fundamental solution, the zero singular value moves to another degenerate scale ($1.05e^{-1}$) instead of original one (1.05) as shown in Fig.3(a). To investigate how seriously the rank deficiency behaves, we plot the second minimum singular value versus the scale in Fig.3(b). It indicates that rank is deficient by one only. By employing the hypersingular equation in the dual BEM, it is found that the singular value of $[L]$ matrix for any scale is nonzero as shown in Fig.3(c). In order to avoid hypersingularity, the CHEEF method by collocating one point outside the domain can promote the rank as shown in Fig.3(d).

6. Conclusions

In this paper, the numerical instability for a torsion problem by using the conventional BEM was addressed. Instead of direct searching for the degenerate scale by trial and error, a more efficient technique was proposed to directly obtain the singular case since only one normal scale needs to be computed. To deal with the numerical instability due to the degenerate scale, three approaches, method of adding a rigid body mode, hypersingular formulation and CHEEF method, were successfully applied to remove the zero singular value. Good agreement between the BEM results and the analytical solutions were obtained if the regularization techniques were used. A numerical example of a circular bar with keyway was demonstrated to check the validity.

Acknowledgments

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Table 1 Degenerate scale and torsion rigidity for a circular bar with keyway using different methods.

Torsion rigidity / Scale / Method	**Normal scale** (***a***=**2.0**)	**Normal scale**	**Degenerate scale** (***a***=**1.05**)
Analytical method	12.6488	$\Gamma = \int_B \psi_1(s)\,dB(s) = 1.5539$	0.9609
Direct BEM (UT)	12.5440 (error=0.83%)		1.8712 (error=94.73%)
Direct BEM (LM)	**Regularization techniques are not necessary**	$d = e^{-\frac{1}{\Gamma}} = 0.5254$ **(Expansion ratio)**	0.9530 (error=0.82%)
Adding rigid body term (*c*=1.0)		2×0.5254=1.0508 **(Degenerate scale)**	0.9876 (error=2.78%)
CHEEF technique (20.0, 20.0)			0.9272 (error=3.51%)

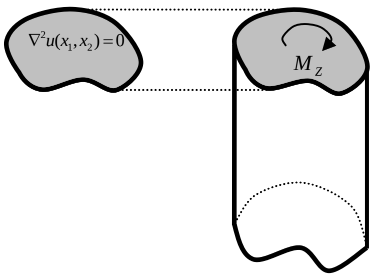

Fig.1 Figure sketch of the torsion problem.

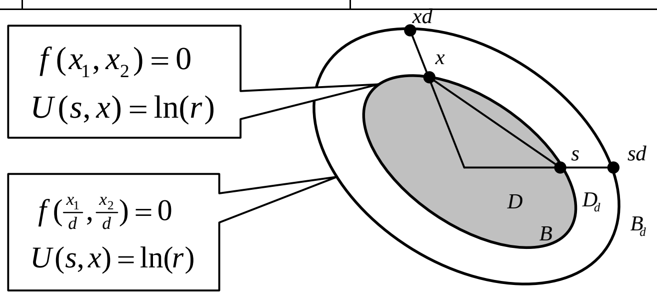

Fig. 2 The normal scale domain D and the degenerate scale domain D_d.

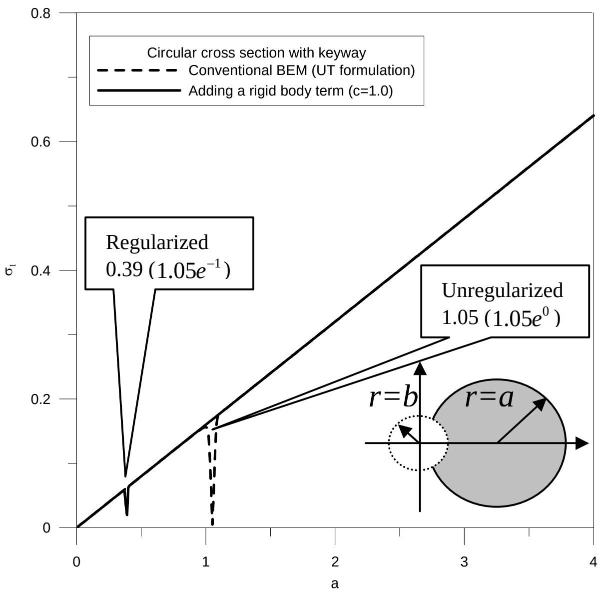

Fig. 3 (a) The first minimum singular value versus scale using the conventional BEM (UT formulation) and method of adding a rigid body term.

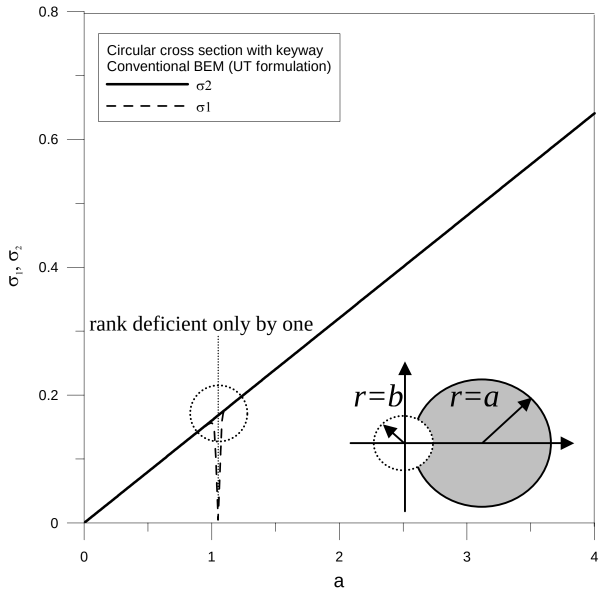

Fig. 3 (b) The second minimum singular value versus scale using the conventional BEM (UT formulation).

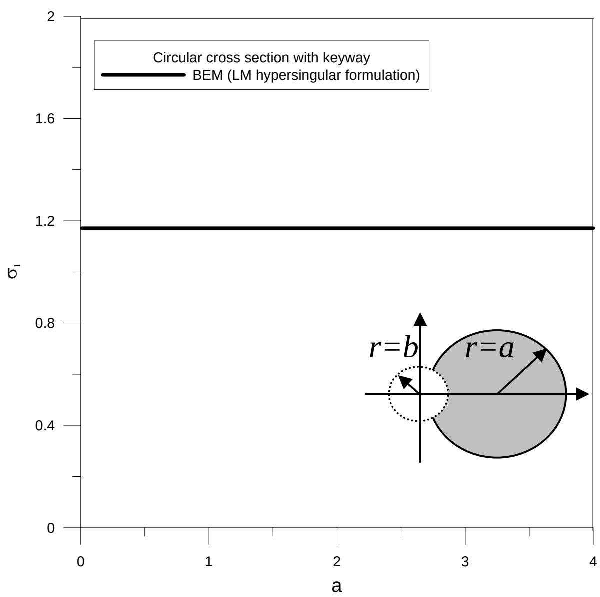

Fig. 3 (c) The first minimum singular value versus scale by using the hypersingular equation (LM formulation).

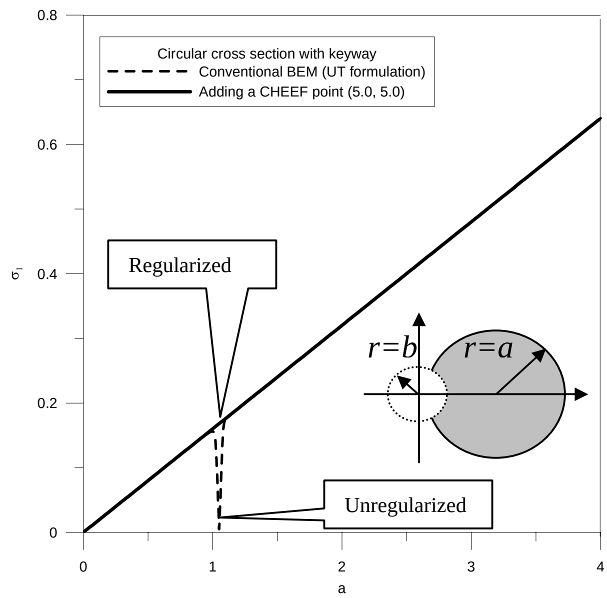

Fig. 3 (d) The first minimum singular value versus scale by using the conventional BEM (UT formulation) and CHEEF method.