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# Dual boundary element analysis for cracked bars under torsion

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## Abstract

A dual integral formulation for a cracked bar under torsion is derived, and a dual boundary element method is implemented. It is shown that as the thickness of the crack becomes thinner, the ill-posedness for the linear algebraic matrix becomes more serious if the conventional BEM is used. Numerical experiments for solution instability due to ill-posedness are shown. To deal with this difficulty, the hypersingular equation of the dual boundary integral formulation is employed to obtain an independent constraint equation for the boundary unknowns. For the sake of computational efficiency, the area integral for the torsion rigidity is transformed into two boundary integrals by using Green's second identity and divergence theorem. Finally, the torsion rigidities for cracks with different lengths and orientations are solved by using the dual BEM, and the results compare well with the analytical solutions and FEM results.

Key words dual BEM, torsion problem, cracked bar, ill-conditioned, condition number, torsion rigidity, degenerate boundary and hypersingular equation.

## Introduction

In studying the torsion problem using numerical techniques, the analyst may encounter problems with singularities. Singular behavior is often ignored in the expectation that the error will be limited to the vicinity of the singularity. However,

it is very important to show how strong the singular behavior is, *e.g.*, the stress intensity factor of fracture mechanics. In finite elements, special singular or hybrid elements are sometimes used instead of the quarter-point rule; *e.g.*, MSC/NASTRAN Version 68 [1] provides the capabilities of singular CRAC3D and CRAC2D elements for crack problems, but the Laplace equation with singularity has not been developed to the author's knowledge. For problems with a degenerate boundary, *e.g.*, crack problems [2, 3, 4, 5, 6, 7], flow around sheet piles [8, 9], and thin airfoil in aerodynamics[10, 11], singularity exists, and the dual integral formulation has been applied successfully. Using the dual integral formulation, all the well-posed boundary value problems can be solved even though a degenerate boundary is present. It is well known that DBEM(dual boundary element) is particularly suitable for the problem of extreme localization and concentration with singularity. The DBEM solution is based on the complete formulation of the dual integral equations. The long standing abstruseness of the nonunique problem in BEM has been solved, and the general purpose program of BEPO2D(Boundary Element POTential 2-D) has been implemented [11].

In this paper, the dual BEM is extended to solve a problem of a cracked bar under torsion. The condition number versus the finite thickness of the crack will show the ill-conditioned behavior using conventional BEM. Numerical experiments for solution instability due to ill-posedness will be shown. For the sake of computational efficiency, the area integral formula for torsion rigidity will be transformed into two boundary integrals. Finally, several examples will be solved using two software packages, the BEPO2D and SDRC-IDEAS programs using DBEM and FEM, respectively. The results will be compared with analytical solutions to assess their accuracy. Also, the torsion rigidities for cracks with different lengths and orientations will be discussed.

## **Formulation for torsion problems of a cracked bar**

The problem of a cracked bar under torsion can be formulated as a Poisson

equation as follows [12, 13]:

$$\nabla^2 \Psi(x, y) = -2, \quad (x, y) \in D, \quad (1)$$

where  $\Psi$  is the torsion(Prandtl) function,  $\nabla^2$  is the Laplacian operator and  $D$  is the domain. The boundary condition is

$$\Psi(x, y) = 0, \quad (x, y) \text{ on the boundary } B = S + C^+ + C^-, \quad (2)$$

where  $S$  is the normal boundary, and  $C^+$  and  $C^-$  are the crack boundaries as shown in Fig.1. Since Eq.(1) contains the body source term, the problem can be reformulated as

$$\nabla^2 \Psi^*(x, y) = 0, \quad (x, y) \in D \quad (3)$$

and the boundary condition is changed to

$$\Psi^*(x, y) = (x^2 + y^2)/2, \quad (x, y) \text{ on the boundary } B \quad (4)$$

where  $\Psi^* = \Psi + \tilde{\Psi}$  and  $\tilde{\Psi} = (x^2 + y^2)/2$ .

This new problem of Eq.(3) can be used to find the solution for a Laplace equation with Dirichelet data in Eq.(4), which is very easy to implement using DBEM. (*e. g.*, the BEPO2D program is used in this study).

The torsion function  $\Psi$  can be obtained from  $\Psi^*$  by superimposing  $\tilde{\Psi}$ , and the torque can then be determined by

$$M_z = \iint_A (x\tau_{yz} - y\tau_{xz}) dx dy, \quad (5)$$

where  $\tau_{yz}$  and  $\tau_{xz}$  are the shearing stress determined by  $\tau_{yz} = -\alpha G \frac{\partial \Psi}{\partial x}$  and  $\tau_{xz} = \alpha G \frac{\partial \Psi}{\partial y}$ ,  $A$  is the area of the cross section,  $G$  is the shear modulus and  $\alpha$  denotes the twist angle per unit length.

By employing the Green's second identity and Eq.(1), the area integral in Eq.(5) can be transformed into a boundary integral and an area integral as follows:

$$M_z = \iint_A (x\tau_{yz} - y\tau_{xz}) dx dy$$

$$\begin{aligned}
&= -\alpha G \iint_A \left( x \frac{\partial \Psi}{\partial x} + y \frac{\partial \Psi}{\partial y} \right) dx dy \\
&= -\alpha G \iint_A (\nabla \tilde{\Psi} \cdot \nabla \Psi) dx dy \\
&= -\alpha G \iint_A \nabla \cdot (\tilde{\Psi} \nabla \Psi) dx dy + \alpha G \iint_A \tilde{\Psi} \nabla^2 \Psi dx dy \\
&= -\alpha G \oint_B \tilde{\Psi} \frac{\partial \Psi}{\partial n} dB - \alpha G \iint_A (x^2 + y^2) dx dy. \tag{6}
\end{aligned}$$

The induced area integral of the second term on the right hand side of the equal sign in Eq.(6) can be reformulated into a boundary integral again by using the Gauss theorem as follows:

$$\begin{aligned}
-\alpha G \iint_A (x^2 + y^2) dx dy &= \frac{-\alpha G}{16} \iint_A \nabla^2 \{(x^2 + y^2)^2\} dx dy \\
&= \frac{-\alpha G}{16} \oint_B \frac{\partial \{(x^2 + y^2)^2\}}{\partial n} dB. \tag{7}
\end{aligned}$$

## Dual boundary element analysis for a cracked bar

The torsion problem can be simulated by using the mathematical model of the Laplace equation as shown in Eq.(3). Now, we will consider the boundary integral formulation for numerical analyses. For the classical Laplace problem in Fig.1, the equations may be generally described as follows:

Governing equation:

$$\nabla^2 \Psi^*(x) = 0, \quad x \text{ in } D. \tag{8}$$

Boundary conditions:

$$\Psi^*(s) = f(s), \quad s \text{ on } B_1 \tag{9}$$

$$\frac{\partial \Psi^*(s)}{\partial n_s} = g(s), \quad s \text{ on } B_2, \tag{10}$$

where  $f(s)$  and  $g(s)$  denote the known boundary data,  $B_1$  and  $B_2$  are the boundaries and  $n_s$  is the normal vector on the boundary point  $s$ . In the torsion problem of Eqs.(3) and (4), no Neumann boundary condition on  $B_2$  is present.

Using the Green's identity, the first equation of the dual regular boundary integral equations for the domain point  $x$  can be derived as follows:

$$2\pi\Psi^*(x) = \int_B T(s, x)\Psi^*(s)dB(s) - \int_B U(s, x)\frac{\partial\Psi^*(s)}{\partial n_s}dB(s), \quad (11)$$

where

$$U(s, x) = \ln(r) \quad (12)$$

$$T(s, x) = \frac{\partial U(s, x)}{\partial n_s}, \quad (13)$$

in which  $r$  is the distance between the field point  $x$  and the source point  $s$ . After taking the normal derivative of Eq.(11), the second equation of the dual regular boundary integral equations for the domain point  $x$  can be derived:

$$2\pi\frac{\partial\Psi^*(x)}{\partial n_x} = \int_B M(s, x)\Psi^*(s)dB(s) - \int_B L(s, x)\frac{\partial\Psi^*(s)}{\partial n_s}dB(s), \quad (14)$$

where

$$L(s, x) = \frac{\partial U(s, x)}{\partial n_x} \quad (15)$$

$$M(s, x) = \frac{\partial^2 U(s, x)}{\partial n_x \partial n_s}, \quad (16)$$

in which  $n_x$  is the normal vector for the field point  $x$ . Eqs.(11) and (14) are termed dual regular boundary integral equations for the domain point  $x$ . The explicit form of the kernel functions can be found in [11]. By tracing the field point  $x$  to the boundary, the dual singular boundary integral equations for the boundary point  $x$  can be derived:

$$\pi\Psi^*(x) = C.P.V. \int_B T(s, x)\Psi^*(s)dB(s) - R.P.V. \int_B U(s, x)\frac{\partial\Psi^*(s)}{\partial n_s}dB(s) \quad (17)$$

$$\pi\frac{\partial\Psi^*(x)}{\partial n_x} = H.P.V. \int_B M(s, x)\Psi^*(s)dB(s) - C.P.V. \int_B L(s, x)\frac{\partial\Psi^*(s)}{\partial n_s}dB(s), \quad (18)$$

where *R.P.V.*, *C.P.V.* and *H.P.V.* denote the Riemann principal value, Cauchy principal value and Hadamard or Mangler principal value, respectively.

Eqs.(17) and (18) are called dual singular boundary integral equations for the boundary point  $x$ . It must be noted that Eq.(18) can be derived just by applying the operator of the normal derivative to Eq.(17). Differentiation of the Cauchy principal value should be carried out carefully using Leibnitz' rule, and then the finite part can be obtained. The finite part has been termed the Hadamard principal value in fracture mechanics [2] or Mangler's principal value in aerodynamics [14]. The commutative property provides us with two alternatives to calculate the Hadamard principal value analytically [2].

The boundary  $B$  contains two parts, the normal boundary  $S$  and degenerate boundary,  $C^+ + C^-$ , as shown in Fig.1 and Eq.(3).

For  $x \in S$ , Eqs.(17) and (18) can be rewritten as

$$\begin{aligned} \pi\Psi^*(x) &= C.P.V. \int_S T(s, x)\Psi^*(s)dB(s) - R.P.V. \int_S U(s, x)\frac{\partial\Psi^*(s)}{\partial n_s}dB(s) \\ &+ \int_{C^+} T(s, x)\Delta\Psi^*(s)dB(s) - \int_{C^+} U(s, x)\Sigma\frac{\partial\Psi^*(s)}{\partial n_s}dB(s) \end{aligned} \quad (19)$$

$$\begin{aligned} \pi\frac{\partial\Psi^*(x)}{\partial n_x} &= H.P.V. \int_S M(s, x)\Psi^*(s)dB(s) - C.P.V. \int_S L(s, x)\frac{\partial\Psi^*(s)}{\partial n_s}dB(s) \\ &+ \int_{C^+} M(s, x)\Delta\Psi^*(s)dB(s) - \int_{C^+} L(s, x)\Sigma\frac{\partial\Psi^*(s)}{\partial n_s}dB(s). \end{aligned} \quad (20)$$

For  $x \in C^+$ , the equations can be expressed as

$$\begin{aligned} \frac{\pi}{2}(\Sigma\Psi^*(x) + \Delta\Psi^*(x)) &= C.P.V. \int_{C^+} T(s, x)\Delta\Psi^*(s)dB(s) - R.P.V. \int_{C^+} U(s, x)\Sigma\frac{\partial\Psi^*(s)}{\partial n_s}dB(s) \\ &+ \int_S T(s, x)\Psi^*(s)dB(s) - \int_S U(s, x)\frac{\partial\Psi^*(s)}{\partial n_s}dB(s) \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{\pi}{2}(\Sigma\frac{\partial\Psi^*(x)}{\partial n_x} + \Delta\frac{\partial\Psi^*(x)}{\partial n_x}) &= H.P.V. \int_{C^+} M(s, x)\Delta\phi(s)dB(s) - C.P.V. \int_{C^+} L(s, x)\Sigma\frac{\partial\phi(s)}{\partial n_s}dB(s) \\ &+ \int_S M(s, x)\Psi^*(s)dB(s) - \int_S L(s, x)\frac{\partial\Psi^*(s)}{\partial n_s}dB(s), \end{aligned} \quad (22)$$

where the sum and difference for the boundary data on  $C$  are defined by

$$\Sigma\Psi^*(s) = \Psi^*(s^+) + \Psi^*(s^-) \quad (23)$$

$$\Delta\Psi^*(s) = \Psi^*(s^+) - \Psi^*(s^-) \quad (24)$$

$$\Sigma\frac{\partial\Psi^*}{\partial n}(s) = \frac{\partial\Psi^*}{\partial n}(s^+) + \frac{\partial\Psi^*}{\partial n}(s^-) \quad (25)$$

$$\Delta\frac{\partial\Psi^*}{\partial n}(s) = \frac{\partial\Psi^*}{\partial n}(s^+) - \frac{\partial\Psi^*}{\partial n}(s^-). \quad (26)$$

Eqs.(23) ~ (26) tell us that the number of unknowns on the degenerate boundary doubles; therefore, the additional hypersingular integral equation of Eq.(22) is necessary. The dual boundary integral equations for the boundary point provide the complete constraints of all the boundary data for the well-posed boundary value problem. It must be noted that the compatible relations of boundary data are dependent no matter whether  $x$  is on  $C^+$  or on  $C^-$  in Eqs.(21) and (22), respectively. Eq.(21) has the same equation for  $x$  on  $C^+$  or  $C^-$ , but Eq.(22) has equations with different signs which are also linearly dependent on each other for  $x$  on  $C^+$  and  $C^-$ . Nevertheless, Eqs.(21) and (22) for  $x$  on  $C^+$  or  $C^-$  are linearly independent for the unknowns on the two sides of the degenerate boundary. Hence, Eq.(22) plays the most important role in a problem with a degenerate boundary.

## Dual boundary element discretization and the closed-form integral formulae for the kernel functions

After deriving the above compatible relationships of the boundary data in Eqs.(17) and (18), the boundary integral equations can be discretized by using constant elements, and the resulting algebraic system can be obtained as

$$[\bar{T}_{ij}]\{\Psi^*\}_j = [U_{ij}]\left\{\frac{\partial\Psi^*}{\partial n}\right\}_j \quad (27)$$

$$[M_{ij}]\{\Psi^*\}_j = [\bar{L}_{ij}]\left\{\frac{\partial\Psi^*}{\partial n}\right\}_j, \quad (28)$$

where  $[ \ ]$  denotes a square matrix,  $\{ \ }$  a column vector and the elements of the square matrices are

$$U_{ij} = R.P.V. \int U(s_j, x_i)dB(s_j) \quad (29)$$



$$\bar{T}_{ij} = -\pi\delta_{ij} + C.P.V. \int T(s_j, x_i)dB(s_j) \quad (30)$$

$$\bar{L}_{ij} = \pi\delta_{ij} + C.P.V. \int L(s_j, x_i)dB(s_j) \quad (31)$$

$$M_{ij} = H.P.V. \int M(s_j, x_i)dB(s_j). \quad (32)$$

All the above formulae can be integrated analytically; the quadrature rule is used to check the regular integral only. The closed-form solutions of Eqs.(29) ~ (32) are summarized below.

First, the components of the normal vectors,  $\mathbf{n}(x)$  and  $\mathbf{n}(s)$ , are defined in the following form, respectively:

$$n_1(s) = \sin(\theta), \quad n_2(s) = -\cos(\theta) \quad (33)$$

$$\bar{n}_1(x) = \sin(\phi), \quad \bar{n}_2(x) = -\cos(\phi), \quad (34)$$

where  $\theta$  and  $\phi$  are shown in Fig.2(a). Then, the inner and cross products are

$$\mathbf{n}(x) \cdot \mathbf{n}(s) = \cos(\phi - \theta) = \cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta) = n_2\bar{n}_2 + n_1\bar{n}_1 \quad (35)$$

$$\mathbf{n}(s) \times \mathbf{n}(x) \cdot \mathbf{e}_k = \sin(\phi - \theta) = \sin(\phi)\cos(\theta) - \cos(\phi)\sin(\theta) = -\bar{n}_1n_2 + \bar{n}_2n_1. \quad (36)$$

Using the following transformation as shown in Fig.2(b), we have

$$\begin{Bmatrix} x_r \\ y_r \end{Bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{Bmatrix} x_1 - s_1 \\ x_2 - s_2 \end{Bmatrix}. \quad (37)$$

For the regular element, the integral formulae is

$$U_{ij} = v \log \sqrt{v^2 + y_r^2} - v + y_r \tan^{-1}(v/y_r) \Big|_{v=-0.5L_j-x_r}^{v=0.5L_j-x_r} \quad (38)$$

$$T_{ij} = \tan^{-1}(v/y_r) \Big|_{v=-0.5L_j-x_r}^{v=0.5L_j-x_r} \quad (39)$$

$$L_{ij} = -\cos(\phi - \theta)\tan^{-1}(v/y_r) - 0.5 \sin(\phi - \theta)\log(v^2 + y_r^2) \Big|_{v=-0.5L_j-x_r}^{v=0.5L_j-x_r} \quad (40)$$

$$M_{ij} = \cos(\phi - \theta) \left[ \frac{\tan^{-1}(v/y_r)}{y_r} + \frac{v}{v^2 + y_r^2} \right] - y_r \sin(\phi - \theta) \frac{1}{v^2 + y_r^2} - \cos(\phi - \theta) \frac{\tan^{-1}(v/y_r)}{y_r} \Big|_{v=-0.5L_j-x_r}^{v=0.5L_j-x_r}, \quad (41)$$

where  $L_j$  is the length of the  $j^{th}$  element.

In calculating the above limiting values ( $x_r \rightarrow 0, y_r \rightarrow 0$ ) for the singular element, the l'Hospital rule and inverse triangular relations should be considered as follows:

$$\tan^{-1}(x) + \tan^{-1}(1/x) = \pi/2 \quad (42)$$

$$\lim_{x \rightarrow 0} \frac{x}{\tan^{-1}(x)} = 1. \quad (43)$$

The closed-form integration for the singular element has the tangent and normal properties of the classical potential theory by way of the  $\sin(\phi - \theta)$  and  $\cos(\phi - \theta)$  behavior in Eq.(41).

To determine the torsion rigidity using Eq.(6), the following boundary integral can be integrated analytically as follows:

$$\oint_B \tilde{\Psi} \frac{\partial \Psi}{\partial n} dB = \oint_B \tilde{\Psi} \frac{\partial \Psi^*}{\partial n} dB - \oint_B \tilde{\Psi} \frac{\partial \tilde{\Psi}}{\partial n} dB = \sum_{j=1}^{N_t} D_j \left( \frac{\partial \Psi^*}{\partial n} \right)_j - \sum_{j=1}^{N_t} F_j, \quad (44)$$

where  $N_t$  is the total number of boundary elements,  $(\frac{\partial \Psi^*}{\partial n})_j$  is the normal gradient of  $\Psi^*$  for the  $j^{th}$  element, and

$$D_j = \oint_{B_j} \tilde{\Psi} dB = \frac{1}{24} L_j^3 + \frac{1}{2} (s_{1j}^2 + s_{2j}^2) L_j \quad (45)$$

$$F_j = D_j (s_{1j} \sin(\theta) - s_{2j} \cos(\theta)), \quad (46)$$

in which  $(s_{1j}, s_{2j})$  is the center coordinate of the  $j^{th}$  element and  $B_j$  is the boundary of the  $j^{th}$  element.

To find the contribution of the contour integral around the the crack tip as shown in Fig.3, we have

$$\lim_{\epsilon \rightarrow 0} \left\{ -\alpha G \int_0^\pi \frac{1}{2} (\epsilon/2)^2 \frac{1}{\sqrt{\epsilon/2}} \epsilon d\theta \right\} = \lim_{\epsilon \rightarrow 0} \left\{ -\frac{\alpha}{2} G \pi (\epsilon/2)^{5/2} \right\} = 0 \quad (47)$$

after using the order of  $\frac{\partial \Psi}{\partial n}$  approaching  $\frac{1}{\sqrt{\epsilon/2}}$  near the crack tip.

Another boundary integral in Eq.(7) can be integrated as follows:

$$\oint_B \frac{\partial \{(x^2 + y^2)^2\}}{\partial n} dB = \sum_{j=1}^{N_t} E_j, \quad (48)$$

where

$$E_j = 8D_j(s_{1j}\sin(\theta) - s_{2j}\cos(\theta)). \quad (49)$$

According to Eqs.(46) and (49), the integrals along the crack surfaces vanish automatically; *i.e.*, the variation of the torsion rigidity due to a crack is only influenced by  $D_j$ .

### **Ill-posedness and its regularization for modeling the degenerate boundary using BEM**

To simulate the zero-thickness crack for the bar, a limiting process with a finite thickness,  $\epsilon$ , is often employed if a convergent solution can be obtained. In this section, it will be proved that this concept fails using the conventional BEM (UT or LM only) numerically. Figs.4 and 5 show the influence of  $\epsilon$  on the torsion function along  $y = 0$  for the torsion problem by using the conventional BEM, UT and LM methods, respectively. As  $\epsilon$  approaches  $0^+$ , the torsion function in the interior points along the crack does not converge to the exact solution. This numerical error occurs even when the boundary point spacing is very small ( $\epsilon \rightarrow 0$ ). This could easily lead the analyst to believe that he is approaching the correct solution even when his answers actually contain a large error. The cause of the error in the preceding solution is the ill-conditioning of the coefficient matrix instead of the singularity of the crack tip. The ill-conditioned behavior depends on the thickness of the crack as shown in Fig.6 using UT only, LM only or dual methods. It is found that the condition number of dual BEM is less than that of the two methods, UT and LM, respectively. As the thickness becomes zero, the algebraic equation leads to a singular matrix using UT or LM only. The difficulty with an infinitely thin crack has been solved by dividing the region into two separate zones, the so-called multi-zone method [16, 17, 18]. The drawback of the multi-zone method is obvious in that the introduction of artificial boundaries is arbitrary, nonunique, and thus not qualified as an automatic scheme. Also, it results in a larger system of equations than is needed. Instead of employing

the multi-zone method, dual BEM can be applied to avoid the the ill-conditioned problem as shown in Fig.6. If the thickness of the crack approaches zero but does not equal zero, the order of dependence for the constraint equations by collocating the boundary points on the two sides of the crack is high using the UT or LM methods. To obtain more independent equations on the two sides of the crack boundary, dual BEM must be considered; *i.e.*, one is the UT equation for one side of the crack boundary, and the other is the LM equation for the other side of the crack boundary. When the thickness of the crack is exactly zero, the method combining the UT and LM equations is called dual BEM. Also, dual BEM can be applied to solve the problem when the finite thickness is near zero. Since the constraint equations obtained by collocating the points on the normal boundary have two choices, the UT or LM equations as shown in Figs.7(a) and 8(a), the two results are satisfactory as shown in Figs.7(b) and 8(b), respectively.

## Results and discussion of analytical and numerical solutions

The analytical solutions for the torsion rigidities of different length direct cracks are shown below [19]:

(1). For the case of  $\frac{a}{d} = 0.50$ , where  $a$  is the crack length and  $d$  is the diameter of the torsion bar, we have the torsion function in polar coordinate  $(r, \phi)$  and torsion rigidity ( $M_z$ ) as shown below:

$$\Psi(r, \phi) = 32 \frac{a^2}{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{r}{a}\right)^{(2n+1)/2} - \left(\frac{r}{a}\right)^2}{(2n+1)[16 - (2n+1)^2]} \sin \frac{(2n+1)\phi}{2} \quad (50)$$

$$M_z = Ga^4 \left\{ \frac{512}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2(2n+5)[16 - (2n+1)^2]} - \frac{\pi}{2} \right\} = 0.878Ga^4 \quad (51)$$

(2). For the case of  $\frac{a}{d} = 0.25$ , the torsion function in bipolar coordinate  $(\alpha, \beta)$  can be expressed as

$$u(\alpha, \beta) = a^2 \sinh^2 \alpha_0 \left\{ -\frac{\sin^2 \beta}{(\cosh \alpha + \cos \beta)^2} \frac{1}{\pi} \sum_{n=0}^{\infty} a_n e^{-(n+\frac{1}{2})(\alpha-\alpha_0)} \sin\left(n + \frac{1}{2}\right)\beta \right\}, \quad (52)$$

where

$$a_n = \int_0^{2\pi} \frac{\sin^2 \beta \sin(n + \frac{1}{2})\beta}{(\cosh \alpha_0 + \cos \beta)^2} d\beta \quad (53)$$

$$\sinh \alpha_0 = \frac{a(d-a)}{d(\frac{1}{2}d-a)}, \quad (a < \frac{d}{2}). \quad (54)$$

The torsion rigidity ( $M_z$ ) is obtained as

$$M_z = Ga^4 \sinh^4 \alpha_0 \left[ \frac{2}{\pi} \sinh \alpha_0 \sum_{n=0}^{\infty} a_n b_n - \frac{1}{2\pi} \sum_{n=0}^{\infty} (2n+1) a_n^2 - \frac{\pi}{2 \sinh^4 \alpha_0} \right], \quad (55)$$

where

$$b_n = \int_0^{2\pi} \frac{\sin^2 \beta \sin(n + \frac{1}{2})\beta}{(\cosh \alpha + \cos \beta)^3} d\beta. \quad (56)$$

After some lengthy calculations, we have

$$M_z = 1.28Ga^4.$$

(3). For the case of no crack,  $\frac{a}{d} = 0$ , we have the torsion function in the  $(x, y)$  coordinate as shown below:

$$\Psi(x, y) = \frac{1}{2}(x^2 + y^2 - \frac{d^2}{4}).$$

The torsion rigidity is found to be

$$M_z = \frac{\pi}{32} Gd^4 \theta.$$

For the case of  $a/d = 0.5$ , the values of the torsion function along  $x = 0$  and  $y = 0$  are those shown in Figs.9(a) and 9(b), respectively. It is found that the results of dual BEM agree with those of the analytical solution [19] and conventional BEM [13]. According to the two boundary integrals in Eqs.(6) and (7) for torsional rigidity, the nondimensional torsional rigidity is that shown in Fig.10 for a radial crack with different lengths. It is found that the torsion rigidities for  $a/d = 0, 0.25$  and  $0.5$  match the analytical solutions in [19]. Also, the results of dual BEM are better than the FEM solutions using the SDRC-IDEAS software.

Similarly, by changing the orientations of the crack angle  $\theta$  as shown in Fig.11 (a), we can construct the curve for  $M_z$  versus  $\theta(0^\circ \sim 45^\circ)$  for a slant crack as shown in Fig.11 (b). Although no analytical solutions can be used for comparison, the DBEM results agree well with the FEM solutions.

## Conclusions

The dual boundary element method has been applied to solve the torsion problems for a cracked bar. The results show that DBEM provides high solution accuracy and greatly simplifies the modeling. DBEM involves modeling only on the boundary without introducing the artificial boundary as the multi-zone method proposes even though the thickness of the crack is zero. Also, the area integral for torsion rigidity is transformed into two boundary integrals by using the Green's second identity and Gauss theorem; therefore, the domain cell does not need to be discretized. The results have been compared with analytical and FEM solutions, and found to be in good agreement.

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## Figure captions

**Fig.1** A cracked torsion bar under torsion.

**Fig.2** (a). Boundary element discretization (b). Coordinate transformation (c). Boundary element mesh.

**Fig.3** Contour integration around the crack tip.

**Fig.4** UT method of solution for the torsion function by means of a limiting process of  $\epsilon = 0.1, 0.01, 0.001$  for the crack thickness.

**Fig.5** LM method of solution for the torsion function by means of a limiting process of  $\epsilon = 0.1, 0.01, 0.001$  for the crack thickness.

**Fig.6** Condition number using the UT, LM and dual BEM(UT combined LM) methods for different crack thicknesses of  $\epsilon = 0.1, 0.01, 0.001$ .

**Fig.7** (a). Constraint equations obtained, (b). dual BEM solution by UT method(mixed with LM method).

**Fig.8** (a). Constraint equations obtained, (b). dual BEM solution by LM method(mixed with UT method).

**Fig.9** (a). Airy stress function along  $x = 0$ , (b). Airy stress function along  $y = 0$

**Fig.10** Torsion rigidities for cracks of different lengths.

**Fig.11** (a). A torsion bar with a slant crack (b). Torsion rigidities for different angles of crack orientation.