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A New Point of View for Householder Orthogonal Matrix by Using Exponential Matrix

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Abstract. There are many different methods to obtain an orthogonal matrix. Householder discovered an orthogonal matrix, and it can be applied to solve physical problems of reflection. In this classroom note, we can obtain an orthogonal matrix group e^{At}, where A is antisymmetric with angular vector w. When t is setting to be $t=p/|w|$, e^{At} is equivalent to a Householder matrix of a reflecting plane with normal vector w.

Key Words. Householder matrix, orthogonal matrix group, residue theorem of matrix function

1. Introduction to Householder Matrix. Orthogonal matrices are very popular not only in mathematics but also in engineering. In mathematics, orthogonal matrix play an important role in linear algebra and matrix theory. In engineering practice, orthogonal matrix form the basis for rigid body dynamics [1,2] and constitutive law for finite deformation. [3-7] How to construct an orthogonal matrix is our concern.

Matrix exponentials are always encountered in biological, physical, economical processes. Many approaches to calculate the matrix exponential were developed. Chen developed the residue theorem for matrix [10] and calculated the matrix exponential efficiently in conjunction with the Cayley-Hamilton theorem.[9] Also, the Householder matrix can be treated as special case of present formulation.

In this note, the matrix exponential e^{At} is derived using anti-symmetric matrix for A to construct the orthogonal matrices.

1.1. Review of Householder Matrix. If nÎR^n be nonzero. An n-by-n matrix H of the form $H=I-2nn^T/n^Tn$ is called a Householder Reflection. It is vertified that Householder matrices are symmetric and orthogonal. A variation of the Householder formula is $H_n(\boldsymbol{u})=\boldsymbol{u}-2(nn^T/n^Tn)\boldsymbol{u}$, that is, $H_n(\boldsymbol{u})=[I-2nn^T/n^Tn]\boldsymbol{u}$. So, we refer to the matrix $H_n=I-2nn^T/n^Tn$ as the Householder matrix and to the linear transformation $H_n(\boldsymbol{u})=\boldsymbol{u}-2(nn^T/nn)\boldsymbol{u}$ as the Householder reflection corresponding to n.

From computation, we know that the reflections H_n have many interesting properties, some of which are as follows:

(1) $H_n=I-2nn^T/n^Tn$, since $H_v=I-2(nn^T/n^Tn)$.

(2) $(H_n)^2=I$, and $H_n^{-1}=H_n$.

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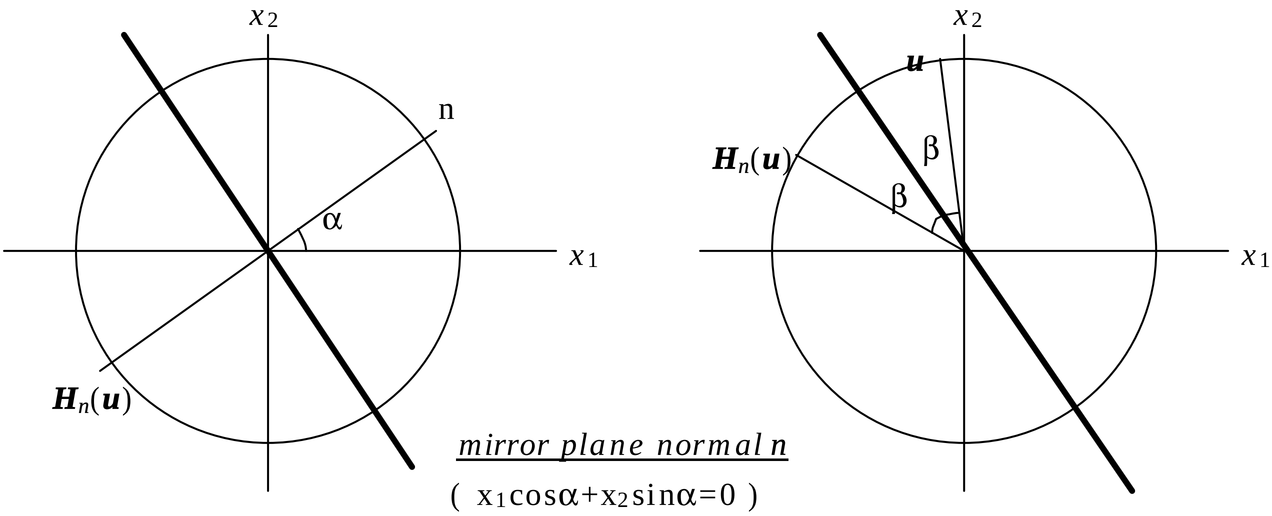
(3) H_n is an orthogonal linear transformation. Since $H_n=H_n^{-1}$, it follows that $m(H_n)=m(H_n)^{-1}=m(H_n)^{T}$, where $m \in R$. So, all reflection matrices are symmetric.

(4) Based on congruence transform, if A is an orthogonal matrix, we can obtain $H^TAH=D$, where D is an congruent matrix.

(5) For n=2, α∈R, $n=\begin{pmatrix} \cos a \\ \sin a \end{pmatrix}$, we can obtain

$$H_n=I-2nn^T/\ |n|^2=\begin{pmatrix} -\cos 2a & -\sin 2a \\ -\sin 2a & \cos 2a \end{pmatrix} \qquad (1)$$

, where $\boldsymbol{u} \in R^2$.



(Fig. 1)

(6) If n=3 and we let $n=\begin{Bmatrix} p \\ q \\ r \end{Bmatrix}$ column vector be a unit vector with $p^2+q^2+r^2=1$.

Then we can obtain

$$H=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-2\begin{Bmatrix} p \\ q \\ r \end{Bmatrix}\{p \quad q \quad r\}=\begin{pmatrix} 1-2p^2 & -2pq & -2qr \\ -2pq & 1-2q^2 & -2qr \\ -2pr & -2qr & 1-2r^2 \end{pmatrix} \qquad (2)$$

1.2. Computation e^{At} by residue theorem. Mathematical models of many mechanical, biological and physical processes involve systems of linear differential equation,

$$\dot{x} = A\boldsymbol{x}, \quad \boldsymbol{x}(0)=\boldsymbol{x}_0 \tag{3}$$

where $\boldsymbol{x}$ is an unknown vector, $\dot{\boldsymbol{x}}$ means the time differential of $\boldsymbol{x}$, $\boldsymbol{x}_0$ is initial condition and A is a given n-by-n matrix. If A is not function of time, Eq.**(3)** has the general solution,

$$\boldsymbol{x}=e^{At}\boldsymbol{x}_0 \tag{4}$$

Assume that the characteristic equation of A is as follows,

$$P_n \lambda^n + P_{n-1} \lambda^{n-1} + \cdots\cdots + P_1 \lambda + P_o = 0 \tag{5}$$

then, Cayley-Hamilton theorem [5] states,

$$P_n \mathrm{A}^n + P_{n-1} \mathrm{A}^{n-1} + \cdots\cdots + P_1 \mathrm{A} + P_o \mathrm{I} = 0 \tag{6}$$

If A is the special case of antisymmertric [9] and 3-by-3 matrix,

$$A=\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \tag{7}$$

Then A has three distinct eigenvalues [8]: 0, $\sqrt{a^2+b^2+c^2}\, i$,and $-\sqrt{a^2+b^2+c^2}\, i$.

The three eigenvalues satisfy eigen equation,

$$l^3+(a^2+b^2+c^2)\, l^2=0 \tag{8}$$

where l is eigenvalue. Based on the Cayley-Hamilton theorem, we have

$$a(t)A^3+b(t)A^2+c(t)A+d(t)I=0 \tag{9}$$

Then $f(A)$ can be expressed as

$$f(A)=e^{At}=[a(t)A^3+b(t)A^2+c(t)A+d(t)I]\, s(t)+ [p(t)A^2+q(t)A+r(t)I] \tag{10}$$

where $a, b, c, d, p, q, r \in R$. Based on the residue theory for matrix, we can obtain e^{At}, (we can obtain the same result by similar transform action as follows.)

$$e^{At}=\frac{1-cos\,|w|\,t}{|w|^2}A^2+\frac{sin\,|w|\,t}{|w|}A+I \tag{11}$$

If we assume $|w|=1$ and $t=p$, then

$$e^{At}=\begin{bmatrix} 2a^2-1 & 2ab & 2ac \\ 2ab & 2b^2-1 & 2bc \\ 2ac & 2bc & 2c^2-1 \end{bmatrix}=[M] \tag{12}$$

Since M is orthogonal, $-M$ is also orthogonal since $(-M)^T(-M)=I$. Therefore,

$$[-M]=\begin{bmatrix} 1-2a^2 & -2ab & -2ac \\ -2ab & 1-2b^2 & -2bc \\ -2ac & -2bc & 1-2c^2 \end{bmatrix} \tag{13}$$

It is interesting to find that $[-M]=H$ if $a=p$, $b=q$, $c=r$.

2.1. Comparsion of e^{At} and H_n Now we can prove $e^{At}=H$ when $t=p$ and $|w|=1$, and $a=p$, $b=q$, $r=s$. It is found that the n vector in H_n forms the basis for antisymmetric matrix A.

2.2. 2-by-2 matrix. If A is the special case of antisymmetric 2-by-2 matrix.

$$A=\begin{pmatrix} 0 & -w \\ w & 0 \end{pmatrix} \tag{14}$$

Then A has two distinct eigenvalues wi, $-wi$. So, we will obtain e^{At}.

$$e^{At}=\frac{\sin|w|t}{w}A+cos|w|t\ I=\begin{pmatrix} cos\,wt & -sin\,wt \\ sin\,wt & cos\,wt \end{pmatrix} \tag{15}$$

And we assume n=2 in Householder matrix, so $n=\begin{Bmatrix} p \\ q \end{Bmatrix}$.

Then we can obtain

$$H=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-2\begin{Bmatrix} p \\ q \end{Bmatrix}\{p \quad q\}=\begin{pmatrix} 1-2p^2 & -2pq \\ -2pq & 1-2q^2 \end{pmatrix} \tag{16}$$

By setting $p=cos\frac{q}{2}$, $q=sin\frac{q}{2}$, we can find

$$H=\begin{pmatrix} 1-2cos^2\frac{q}{2} & -2cos\frac{q}{2}sin\frac{q}{2} \\ -2cos\frac{q}{2}sin\frac{q}{2} & 1-2\,sin^2\frac{q}{2} \end{pmatrix}=\begin{pmatrix} -cos\,2q & -sin\,2q \\ -sin\,2q & cos\,2q \end{pmatrix} \tag{17}$$

Therefore, we can say that the result is the same as shown in Eq. (1).

2.3. Comparison of 4-by-4 matrix. If A is the special case of antisymmetric and 4-by-4 matrix

$$A=\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \tag{18}$$

Then A has four distinct eigenvalues, $l_1=\dfrac{-\sqrt{-w^2-\sqrt{(w^2+t)(w^2-t)}}}{\sqrt{2}}$,

$l_2=\dfrac{\sqrt{-w^2-\sqrt{(w^2+t)(w^2-t)}}}{\sqrt{2}}$, $l_3=\dfrac{-\sqrt{-w^2+\sqrt{(w^2+t)(w^2-t)}}}{\sqrt{2}}$, and

$$l_4=\frac{\sqrt{-w^2+\sqrt{(w^2+t)(w^2-t)}}}{\sqrt{2}}, \text{ where } w=\sqrt{a^2+b^2+c^2+d^2+e^2+f^2}, \; t=2be\text{-}2cd\text{-}2af$$

Assume $w=\sqrt{a^2+b^2+c^2+d^2+e^3+f^2}=1$, $t=2be\text{-}2cd\text{-}2af=0$, we will obtain e^{At}.

$$e^{At}=(1\text{-}cost)A^2+I \quad (\text{assume } t{=}p) \tag{19}$$

3. Eulerian Angle. Before setting up the motion of rigid bodies [1] in the Lagrangian formulation of mechanics, it will therefore be necessary to seek three independent parameters specifying the orientation of a rigid body. Only when such generalized coordinates are constructed, one can write a Lagrangian for the system and obtain the Lagrangian equations of motion. A number of such sets of parameters have been described in the literatures, but the most common and useful are the Eulerian angles. We shall therefore define these angles at this point, and show how the elements of the orthogonal transformation matrix can be expressed in terms of them. The Eulerian angles are then defined as the three successful angles of rotation. If we want to transformate n to n', we will perform the three successive angles of rotation.

$$\begin{Bmatrix} n'_1 \\ n'_2 \\ n'_3 \end{Bmatrix} = DCB \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \tag{20}$$

The sequence will start by rotating the initial system of axes, *xyz*, by an angle *f* counterclockwise about the *z* axis, and the resultant coordinate system will be labelled the *xhz¢*. In the second stage the intermediate axes, *xhz¢*, are rotated about the *x* axis counterclockwise by an angle *q* to produce another intermediate set. In other words, *xhz¢*, are rotated about the *z¢*axis counterclockwise by an angle *y* to produce another intermediate set. Now the transformation matrix of *D* is a rotation about *z*, and hance has a matrix of the form

$$D=\begin{pmatrix} cos f & sin f & 0 \\ -sin f & cos f & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{21}$$

The *C* matrix corresponds to a rotation about ξ, and can be shown

$$C=\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos q & sin q \\ 0 & -sin q & cos q \end{pmatrix} \tag{22}$$

Finally, B is a rotation about ζ′ and therefore has the same form as D:

$$B=\begin{pmatrix} \cos y & \sin y & 0 \\ -\sin y & \cos y & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{23}$$

3.1. Comparsion with e^{At}. In the special case of antisymmetric 3-by-3 matrix A, we can obtain some orthogonal matrices by choosing special values of a,b,c:

(1) If $a=0$, $b=0$, $c=1$, then $e^{A_1 t}=\begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$. (24)

(2) If $a=1$, $b=0$, $c=0$, then $e^{A_1 t}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}$. (25)

(3) If $a=0$, $b=1$, $c=0$, then $e^{A_1 t}=\begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}$. (26)

So, we know $e^{(A_1+A_2+A_3)t}=DCB$.

For the same reason, we can carry out the transformation from a given Cartesian coordinate system to another by means of three successive rotations by using e^{At}.

3.2. Comparsion by another idea. Since B, C, and D are orthogonal matrices, and the product matrix $Q=BCD$ is, too. It follows as

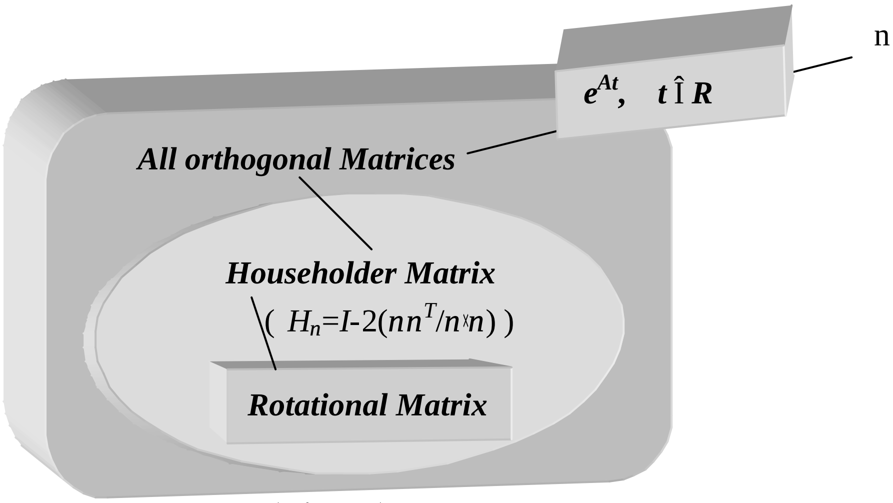
$$Q=\begin{pmatrix} \cos y\cos f-\cos q\sin f\sin y & \cos y\sin f+\cos q\cos f\sin y & \sin y\sin q \\ -\sin y\cos f-\cos q\sin f\cos y & -\sin y\sin f+\cos q\cos f\cos y & \cos y\sin q \\ \sin q\sin f & -\sin q\cos f & \cos q \end{pmatrix} \tag{27}$$

From 1.2., we can calculate e^{At} as

$$e^{At}=\begin{pmatrix} \frac{1-\cos wt}{w^2}\times(-b^2-c^2)+1 & \frac{1-\cos wt}{w^2}\times ab-\frac{\sin wt}{w}\times c & \frac{1-\cos wt}{w}\times ac+\frac{\sin wt}{w}\times b \\ \frac{1-\cos wt}{w^2}\times ab+\frac{\sin wt}{w}\times c & \frac{1-\cos wt}{w^2}\times(-a^2-c^2)+1 & \frac{1-\cos wt}{w^2}\times bc-\frac{\sin wt}{w}\times a \\ \frac{1-\cos wt}{w^2}\times ac-\frac{\sin wt}{w}\times b & \frac{1-\cos wt}{w^2}\times bc+\frac{\sin wt}{w}\times a & \frac{1-\cos wt}{w^2}\times(-a^2-b^2)+1 \end{pmatrix}$$

(28)

4. The relation and graph of e^{At}. The set of orthogonal matrix is shown in Fig. 2.



(Fig. 2)

5. Orthogonal matrix group. The computation of e^{At} is an orthogonal matrix group. When $|w|t=0,\ 2p$, e^{At} will be a unit matrix I. When $|w|t=p$, e^{At} will be a Householder orthogonal matrix which can make any vector reflect to another by a mirror plane with normal vector n. When $|w|t=p/2,\ 3p/2$, it is an another orthogonal matrix. In physical meaning, it can make any vector project to the mirror. Because it rotates half the Householder matrix. So, it serves as a projector. As $|w|t$ varies, e^{At} is an orthogonal matrix and they can transform a vector to another vector are shown in Fig.2.

$|w|t=0 \rightarrow e^{At}= I.$ (29)

$|w|t=p \rightarrow e^{At}=$ *Householder Matrix*. (30)

$|w|t=p/2$ or $3p/2 \rightarrow e^{At}=$ *Projection orthogonal matrix.* (31)

$|w|t=0\sim 2p \rightarrow e^{At}=$ *is another kind of orthogonal matrix.* (32)

So, in conclusion, the e^{At} is a orthogonal matrix group, and all kinds of orthogonal martices are it's special cases.

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