Green's Functions. Lecture 7. Modified Green's Function. Haberman, 9.4.3.

1. **Problem**. Solve $f = L(y) := (py')' + qy$ on $[a, b]$,
$B_1(y) := a_1 y(a) + a_2 y'(a) = 0$,
$B_2(y) := b_1 y(b) + b_2 y'(b) = 0$,
when we assume Assumption 1 fails, there is $\phi_h \neq 0$ with $L(\phi_h) = 0$, $B_1(\phi_h) = 0$, and $B_2(\phi_h) = 0$.

2.**Ex 1**. Let $[a, b] = [0, L]$, $Ly := y'' = f$, $B_1(y) = y'(0)$, $B_2(y) = y'(L)$. Take $\phi_h(x) = 1$ for all x, and suppose $\int_0^L f = 0$. A particular solution is $y_p(x) = \int_0^L G_m(x, s)f(s)ds$, where

$$G_m(x, s) = \begin{cases} -\frac{x^2+s^2}{2L} + s & x < s \\ -\frac{x^2+s^2}{2L} + x & x > s \end{cases} \tag{1}$$

The general solution is $y(x) = y_p(x) + c$ for any real c. We call G_m a modified Green's function.

3. **First step to get a solution**. How to get a modified Green's function. We start by saying that the first version of a modified Green's function $G_m^{(1)}(x, x_0)$ will not be the solution to $L_B(y) = \delta(x - x_0)$, but to $L_B(y) = \delta(x - x_0) - c_1\phi_h$ where the right hand side is orthogonal to ϕ_h, i.e. $\int_a^b [\delta(x - x_0) - c_1\phi_h(x)]\phi_h(x)dx = 0$. Thus we choose $c_1 = c_1(x_0) = \frac{\phi_h(x_0)}{\int_a^b \phi_h^2}$.

4. **HOW TO SOLVE IT** Consider

$$Ly(x) = \delta(x - x_0) - c_1(x_0)\phi_h(x). \tag{2}$$

$$B_1 y = 0. \quad B_2 y = 0. \tag{3}$$

We will work through this method in Ex 2. The way we do Ex 1 in section 8 is special, in parts.

4.1. We find the general solution of (2) on (a, x_0), and on (x_0, b), i.e. the solution of $Ly = -c_1\phi_h(x)$. On each interval, we use variation of parameters to find a particular solution, then add on $N(L)$.

4.2. Now we find the general solution u_L of (2) on (a, x_0), satisfying $B_1(u_L) = 0$. Similarly we find the general solution u_R of (2) on (x_0, b), satisfying $B_2(u_R) = 0$.

4.3. We want a particular solution y_p which is u_L on the left, and u_R on the right, and is continuous at x_0, and $u_R'(x_0) - u_L'(x_0) = 1/p(x_0)$, as in Lecture 4. The general solution to (2) and (3) is $y = y_p + c\phi_h$.

4.4. Next, replacing $G_m^{(1)}(x, x_0)$ by $G_m^{(2)}(x, x_0) = G_m^{(1)}(x, x_0) + c_2\phi_h(x)$, we can get $G_m^{(2)}(x, x_0)$ orthogonal to ϕ_h. We choose $c_2 = -\frac{\int_a^b \phi_h(x)G_m^{(1)}(x,x_0)dx}{\int_a^b \phi_h^2}$. The Theory sections show ${G_m}^{(2)}(x, x_0)$ is symmetric, and this is all we need for the solution of $L_B y = f$ to be $y(x) = \int {G_m}^{(2)}(x, s)f(s)ds$. So in fact instead of constructing $G_m^{(2)}$ we can choose $c_2 = c_2(x_0)$ so that $G_m(x, x_0) = G_m^{(1)}(x, x_0) + c_2\phi_h(x)$ is symmetric. Thus we have a whole lot of chouces of $G_m(x, x_0)$.

5. **Ex. 2**. Find a modified Green's function to solve $Ly := y'' + y = f$, $B_1 y := y(0) = 0, B_2 y := y(\pi) = 0$, for any f such that $\int_0^\pi \sin(s)f(s)ds = 0$. Note $\phi_h(x) = \sin(x)$ is a nonzero solution to $Lu = 0, B_1 u = 0, B_2 u = 0$. We find $G(x, s)$ so that $y(x) = \int_0^\pi G(x, s)f(s)ds$.

5.1 $\int_0^\pi \sin^2(x)dx = \pi/2$, so $c_1 = \frac{\sin(x_0)}{\pi/2}$, and we solve y $L_B y(x) = F(x) := -\frac{2}{\pi}\sin(x_0)\sin(x)$.

5.2. On $[0, x_0]$, and on $[x_0, \pi]$, we solve $Ly = F = -\frac{2}{\pi}\sin(x_0)\sin(x)$, by variation of parameters.

Put $u_1 = \cos$ and $u_2 = \sin$. Solve

$$\begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix} \begin{pmatrix} v_1'(x) \\ v_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

Get

$$\begin{pmatrix} v_1'(x) \\ v_2'(x) \end{pmatrix} = -\frac{2}{\pi}\sin(x_0) \begin{pmatrix} -\sin^2(x) \\ \sin(x)\cos(x) \end{pmatrix}$$

We find the simplest antiderivatives:

$$\begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} = \frac{1}{2\pi}\sin(x_0) \begin{pmatrix} 2x - \sin(2x) \\ \cos(2x) \end{pmatrix}$$

Then $y_p = v_1 u_1 + v_2 u_2 = \frac{-1}{2\pi}\sin(x_0)(\sin(x) - 2x\cos(x))$ (by simplifying) is a particular solution.Since $L(sin) = 0$, we delete the first term for a simpler particular solution, $y_p = \frac{1}{\pi}\sin(x_0)x\cos(x)$, and the general solution is $y = y_p + A\cos(x) + B\sin(x)$.

5.3. We want $y = y_p + A_L\cos(x) + B_L\sin(x)$ on $[0, x_0]$ to satisfy $B_1 y = 0$. Get $A_L = 0$ and so

$$y = y_p + B_L\sin(x). \tag{4}$$

5.4. We want $y = y_p + A_R\cos(x) + B_R\sin(x)$ on $[x_0, \pi]$ to satisfy $B_2 y = 0$. Get $A_R = -\sin(x_0)$ and so

$$y = y_p - \sin(x_0)\cos(x) + B_R\sin(x). \tag{5}$$

5.5. We want y continuous at x_0, so $y_p(x_0) + B_L\sin(x_0) = y_p(x_0) - \sin(x_0)\cos(x_0) + B_R\sin(x_0)$, giving $B_L = B_R - \cos(x_0)$, and we substitute in (4) to give, on $[0, x_0]$,

$$y = y_p - \cos(x_0)\sin(x) + B_R\sin(x). \tag{6}$$

5.6.Note. We have a requirement $y'(x_0+) - y'(x_0-) = 1/p(x_0)$, and this is met. So imposing either continuity or $y'(x_0+) - y'(x_0-) = 1/p(x_0)$ gives the same relationship $B_L = B_R - \cos(x_0)$. By differentiating (5) and(6) and substituting $x = x_0$,

$$y'(x_0+) = y_p'(x_0) + \sin(x_0)\sin(x_0) + B_R\cos(x_0)$$

$$y'(x_0-) = y_p'(x_0) - \cos(x_0)\cos(x_0) + B_R\cos(x_0),$$

so $y'(x_0^+) - y'(x_0^-) = 1$. 5.7. The general solution is

$$y(x) = \frac{1}{\pi}\sin(x_0)x\cos(x) - \begin{Bmatrix} \sin(x_0)\cos(x) & x \geq x_0 \\ \sin(x)\cos(x_0) & x \leq x_0 \end{Bmatrix} + B_R(x_0)\sin(x) \tag{7}$$

5.8. We add $B_R(x_0)\sin(x)$ to make y symmetric, as required by the theory. We can do this by eye.

$$y(x) = +\frac{1}{\pi}\sin(x_0)x\cos(x) + \frac{1}{\pi}\sin(x)x_0\cos(x_0) - \begin{Bmatrix} \sin(x_0)\cos(x) & x \geq x_0 \\ \sin(x)\cos(x_0) & x \leq x_0 \end{Bmatrix} + B\sin(x_0)\sin(x) \tag{8}$$

5.9. Instead of adding $B_R(x_0)\sin(x)$ to make y symmetric by eye, we can take $y + c_2\phi_h$ with c_2 uniquely chosen to make it perpendicular to ϕ_h, which makes it symmetric, and gives $G_m^{(2)}(x, x_0)$.

6. (Theory) **G Symmetric** We claim $G_m(x, x_0) = G_m^{(2)}(x, x_0)$ is symmetric. Proof. Recall Green's formula. Let $L(y) := (py')' + qy$ on $[a, b]$. Then for any u and v,

$$\int_a^b uL(v) - vL(u) = p(uv' - vu')|_a^b = pW(u, v)|_a^b.$$

Put $u = G_m(x, x_1)$ and $v = G_m(x, x_2)$.

$$\int_0^L G_m(x, x_1)[\delta(x - x_2) - c_1(x_2)\phi_h(x)] - G_m(x, x_2)[\delta(x - x_1) - c_1(x_1)\phi_h(x)]$$
$$= p[G_m(x, x_1)G_m'(x, x_2) - G_m(x, x_2)G_m'(x, x_1)|_a^b.$$

Since the $G_m(x, x_i)$ are orthogonal to ϕ_h, the LHS gives only $G_m(x_2, x_1) - G_m(x_1, x_2)$. The RHS is zero, and G_m is symmetric.

7. (Theory) **Get solution, assuming symmetry** Now use Green's formula with $Lu = f$ and $v = G_m(x, x_2)$.

$$\int_0^L u(x)[\delta(x - x_2) - c_1(x_2)\phi_h(x)] - G_m(x, x_2)f(x)\, dx = 0$$

Hence, by writing out $c_1(x_2)$,

$$u(x_2) = \int_0^L G_m(x, x_2)f(x)dx + \frac{\phi_h(x_2)}{\int_0^L \phi_h^2} \int_0^L u(x)\phi_h(x)dx$$

The second term is a multiple of $\phi_h(x_2)$, the solution of $L_B y(x_2) = 0$, and we subtract it to get a solution of $L_B y = f$, and swap x and x_2. Then

$$u(x) = \int_0^L G_m(x, s)f(s)ds.$$

8. **Ex 1 cont**. Let $G = G_m^{(1)}$ be the solution of $y'' = \delta(x - x_0) - c_1\phi_h(x)$. Now from 2, $c_1 = \frac{\phi_h(x_0)}{\int_a^b \phi_h^2} = 1/L$. Then $G'' = -1/L$ on $[0, x_0)$ and on $(x_0, L]$. Hence $G' = -x/L$ for $x < x_0$, since $G_m'(0) = 0$. And $G' = (L - x)/L$ for $x > x_0$, since $G_m'(L) = 0$. Now $G'' = \delta(x - x_0) - 1/L$, so G' has a jump of 1 at x_0. We already have this.

Integrate to give $G_m^{(1)}(x, x_0) = -x^2/(2L) + c_3(x_0)$, $(x < x_0)$. And $G_m^{(1)}(x, x_0) = -x^2/(2L) + x + c_4(x_0)$, $(x > x_0)$. We have continuity at x_0, so $c_3 = x_0 + c_4$.

We use Section 7. $\int_0^L G_m^{(1)}(x, x_0)dx = -L^3/(6L) + c_4(x_0)L + x_0^2 + (L^2 - x_0^2)/2 = L^2/3 + x_0^2/2 + c_4(x_0)L$. From 3, $c_2 = -\int_0^L G_m^{(1)}(x, x_0)dx/L = -L/3 - c_4(x_0) - x_0^2/(2L)$. Then

$$G_m^{(2)}(x, x_0) = G_m^{(1)}(x, x_0) + c_2(x_0)\phi_h(x) = \begin{cases} -\frac{x^2 + x_0^2}{2L} + x_0 - L/3 & x < s \\ -\frac{x^2 + x_0^2}{2L} + x - L/3 & x < s \end{cases}$$

This gives the unique Green's function orthogonal to $\phi_h(x)$. Note we can add on any multiple of $\phi_h(x)\phi_h(x_0)$ and still have a symmetric Green's function giving the same operator on functions f which are orthogonal to ϕ_h.