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A NEW POINT OF VIEW FOR THE HOUSEHOLDER MATRIX BY USING MATRIX EXPONENTIAL

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Abstract: It is well known that many approaches can yield the orthogonal matrix. Householder employed the mirror technique to derive the symmetric orthogonal matrix. The Householder matrices of odd orders are found to have the matrix forms of e^{At} and e^{iBt}, and those of even orders are found to have the matrix forms of e^{iBt}, respectively, where A is an anti-symmetric matrix, B is a symmetric matrix and t is a specified time. Householder matrices with dimension of one by one to five by five are constructed by the proposed formulation. Also, the relation among orthogonal, Householder and Hermitian matrices are discussed.

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1. Introduction

Orthogonal matrices are always encountered not only in mathematics but also in engineering. In mathematics, orthogonal matrices play an important role in linear algebra [17, 19] and matrix theory [1, 5, 6, 8, 11]. In engineering practice, orthogonal matrices are the basis for rigid body

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dynamics and constitutive law for finite deformation [3, 7, 13, 14, 16]. How to construct an orthogonal matrix is of our concern [5, 9, 13].

Matrix exponentials were always used in biological, physical, economical and control processes [4, 9, 10, 12, 18, 20]. Nineteen dubious ways to calculate the matrix exponential were developed [15]. Chen developed the residue theorem for matrix [2] and calculated the matrix exponential efficiently in conjunction with the *Cayley-Hamilton Theorem* [9]. Householder matrix is one kind of symmetric orthogonal matrix which is derived by geometric transformation of mirror mapping. Only a few papers [13] have discussed the relation between Householder matrix and matrix exponential.

In this paper, the matrix exponential e^{At} and e^{iBt} are derived by using an anti-symmetric matrix A and symmetric matrix B to construct the orthogonal matrices. The Householder matrix will be treated as a special case of the matrix exponential. Also, several examples will be demonstrated to check the validity of the present formulation.

2. Review of Householder Matrix

If $\underset{\sim}{\nu} \in \mathbb{R}^n$ is a nonzero vector, an n by n matrix of the form $H = I - \frac{2\underset{\sim}{\nu}\underset{\sim}{\nu}^T}{\underset{\sim}{\nu}^T\underset{\sim}{\nu}}$ is called a Householder matrix. In the references [1, 6, 8], we can find that H has many properties, as shown below:

(a) $H^T = H$ (symmetry).

(b) $H^T H = HH^T = H^2 = I$, where I is an identity matrix.

(c) $H\underset{\sim}{y} = \underset{\sim}{P} \to H(H\underset{\sim}{y}) = \underset{\sim}{y}$, where $\underset{\sim}{y}$ is an arbitrary vector, $\underset{\sim}{P}$ is a mirror-mapped vector by the Householder transformation.

(d) For the special case, it can be shown in Figure 1. Given

$$\underset{\sim}{\nu} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \tag{1}$$

we can obtain $H_\nu = I - \frac{2\underset{\sim}{\nu}\underset{\sim}{\nu}^T}{\underset{\sim}{\nu}^T\underset{\sim}{\nu}}$, where $\underset{\sim}{\nu} \in R^2$ and

$$H_\nu = \begin{bmatrix} -\cos(2\alpha) & -\sin(2\alpha) \\ -\sin(2\alpha) & \cos(2\alpha) \end{bmatrix}. \tag{2}$$

3. Similarity Theorem and Residue Theorem for a Matrix

3.1. Similarity Theorem for Matrix

An n by n matrix A is similar to a matrix D, if and only if C and C^{-1} exist and

$$A = CDC^{-1}. \tag{3}$$

The eigenvalues of A are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$, and the corresponding eigenvectors of A are $\phi_1, \phi_2, \ldots, \phi_n$, respectively. The diagonal matrix with the diagonal elements of eigenvalues is shown below:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}. \tag{4}$$

The modal matrix can be composed by assembling the eigenvectors as shown below:

$$C = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \end{bmatrix}. \tag{5}$$

Eq. (3) is the relation between the similar matrices A and D.

3.2. Residue Theorem for Matrix

Given a real-variable function $f(x)$, it can be divided by $(x - a)$ with the residue, $f(a)$. The formula can be written as

$$f(x) = (x - a)Q(a) + f(a), \tag{6}$$

where $Q(x)$ is a quotient term and $f(a)$ is a residue. If the divided term is an n^{th} order polynomial, we have

$$f(x) = (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)Q(x) + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0. \tag{7}$$

Eq. (7) can be extended to a matrix form [2, 9] as shown below:

$$f(A) = (a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I)Q(A) + r_{n-1}A^{n-1} + \cdots + r_1 A + r_0 I, \tag{8}$$

where $a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I = 0$ stems from the *Cayley-Hamilton Theorem* for the A matrix. If the matrix A has multiple eigenvalues, we must differentiate Eq. (8) because the number of equations is less than that of the undetermined coefficients. For this reason, we should differentiate n-times if the matrix A has n-multiple eigenvalues. Residue Theorem for Matrix is particularly powerful for the case of n-multiple eigenvalues, since it is not necessary to calculate the eigenvectors in advance. We will illustrate this point by using the following two examples.

Example 1. Given a matrix $E = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, we can calculate the matrix exponential

$$e^{Et} = (-6te^{2t} - 3e^{2t} + 4e^{3t})I + (5te^{2t} + 4e^{2t} - 4e^{3t})E + (-te^{2t} - e^{2t} + e^{3t})E^2, \quad (9)$$

where the divided term is zero since it obeys the *Cayley-Hamilton Theorem.*

Based on $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $E^2 = \begin{bmatrix} 4 & 0 & 5 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, we can obtain

$$e^{Et} = \begin{bmatrix} e^{2t} & 0 & e^{3t} - e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

Example 2. Given a matrix $F = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, we can calculate the matrix exponential

$$e^{Ft} = (1 - \lambda t + \frac{\lambda^2 t^2}{2} e^{\lambda t} F + \frac{t^2}{2} e^{\lambda t} F^2), \quad (10)$$

where the divided term is zero since it obeys the *Cayley-Hamilton Theorem.*

Based on $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $F^2 = \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}$, we can obtain

$$e^{Ft} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

4. Relationship Between e^{At} and 3 by 3 Householder Matrix

By setting a column vector $\underset{\sim}{v} = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$, we can calculate the 3 by 3 Householder matrix by

$$H = I - \frac{2\underset{\sim}{v}\underset{\sim}{v}^T}{\underset{\sim}{v}^T\underset{\sim}{v}} = \begin{bmatrix} 1-2a^2 & -2ab & -2ac \\ -2ba & 1-2b^2 & -2bc \\ -2ca & -2cb & 1-2c^2 \end{bmatrix}, \tag{11}$$

where the length of $\underset{\sim}{v}$ is unity, i.e., $\underset{\sim}{v}^T\underset{\sim}{v} = 1$.

By choosing the anti-symmetric matrix $A = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$, the three eigenvalues of A are 0, $\sqrt{a^2+b^2+c^2}\, i$, and $-\sqrt{a^2+b^2+c^2}\, i$. By using the Residue Theorem for Matrix, we can obtain

$$e^{At} = (\frac{1-\cos\omega t}{\omega^2})A^2 + (\frac{\sin\omega t}{\omega})A + I, \tag{12}$$

where $\omega = \sqrt{a^2+b^2+c^2} = 1$. Eq. (12) is also termed Euler-Rodrigues Formula. If t is equal to π, we can obtain

$$e^{A\pi} = \begin{bmatrix} 2a^2-1 & 2ab & 2ac \\ 2ba & 2b^2-1 & 2bc \\ 2ca & 2cb & 2c^2-1 \end{bmatrix}. \tag{13}$$

By comparing Eq. (13) with Eq. (12), we find that the relation between the matrix exponential and the Householder matrix is

$$e^{At} = -H. \tag{14}$$

5. Relationship Between an Odd-order Anti-symmetric Matrix A and e^{At}

For an anti-symmetric matrix A with odd dimension, it is easy to find that AA^T is equal to $-A^2$, where AA^T is positive definite and symmetric. The eigenvalues for the matrix are imaginary numbers and conjugate to each other. Also, zero must be one of the eigenvalues, i.e.,

$$A\underset{\sim}{\nu} = 0 \cdot \underset{\sim}{\nu}, \tag{15}$$

where $\underset{\sim}{\nu}$ is the corresponding eigenvector for the zero eigenvalue. By setting the eigenvalues of A to be $0, \beta_1 i, -\beta_1 i, \beta_2 i, -\beta_2 i, \cdots$, and $\beta_n i, -\beta_n i$, the corresponding eigenvectors are $\underset{\sim}{\nu}, \phi_1, \phi_1^*, \phi_2, \phi_2^*, \cdots, \phi_n$, and ϕ_n^*, where $\underset{\sim}{\nu}$ and $\phi_1, \phi_1^*; \phi_2, \phi_2^*, \cdots, \phi_n$, and ϕ_n^* vectors are orthogonal to each other.

We will elaborate it later on.

By setting $p + iq$ to be the complex eigenvector of A, we have

$$A(p + iq) = i\beta_n(p + iq). \tag{16}$$

By taking the conjugate of Eq. (16), we have

$$A(p - iq) = -i\beta_n(p - iq). \tag{17}$$

If Eq. (16) is prepultiplied by $(p + iq)^T$, we have

$$(p + iq)^T A(p + iq) = i\beta_n(p + iq)^T(p + iq). \tag{18}$$

Eq. (18) can be rearranged as

$$\begin{aligned}(p^T Ap - q^T Aq) + i(p^T Aq + q^T Ap) \\ = \beta_n[-(p^T q + q^T p) + i(p^T p - q^T q)].\end{aligned} \tag{19}$$

By setting $p^T Ap$ to be a real constant η, we have

$$\eta = \eta^T, \tag{20}$$

i.e.,

$$(p^T Ap)^T = p^T A^T p = -p^T Ap. \tag{21}$$

From Eq. (21), we have

$$\eta = -\eta^T. \tag{22}$$

By comparing Eq. (22) with Eq. (20), we can obtain

$$\eta = p^T A p = 0. \tag{23}$$

In a similar way, we can derive

$$q^T A q = 0. \tag{24}$$

Therefore, the real-part of the left hand side in Eq. (19) is zero. By setting $p^T Ap$, and $q^T Ap$ to be real constants, ξ and ζ, respectively, we have

$$\xi^T = \xi, \tag{25}$$

$$\zeta^T = \zeta, \tag{26}$$

$$(p^T Aq)^T = q^T A^T p = -q^T Ap = -\zeta = \xi^T = \xi\,. \tag{27}$$

From Eq. (27), we have

$$\xi = -\zeta, \tag{28}$$

i.e.,

$$p^T Aq = -q^T Ap. \tag{29}$$

By comparing Eq. (29) with Eq. (19), the imaginary-part of Eq. (19) is also zero. Therefore, we have

$$p^T p = q^T q. \tag{30}$$

Since

$$p^T q = q^T p, \tag{31}$$

we have

$$p^T q = q^T p = 0. \tag{32}$$

By setting $p^T p = 1$, we have

$$p^T p = q^T q = 1. \tag{33}$$

Because the matrix, A^2, is a symmetric matrix, we have

$$A^2 = \Phi D_A^2 \Phi^{-1} = \Phi D_A^2 \Phi^T. \tag{34}$$

Therefore, the relation between A and Φ is

$$A\Phi = \Phi D_A, \tag{35}$$

i.e.,

$$A = \Phi D_A \Phi^{-1} = \Phi D_A \Phi^T, \tag{36}$$

where

$$D_A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \beta_1 i & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & -\beta_1 i & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \beta_2 i & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_2 i & \vdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \vdots & \beta_n i & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & -\beta_n i \end{bmatrix}_{(2n+1)\times(2n+1)}, \tag{37}$$

$$\Phi = [\underline{\nu}, \phi_1, \phi_1^*, \phi_2, \phi_2^*, \cdots, \phi_n, \phi_n^*], \tag{38}$$

$$\Phi^T = \Phi^{-1} = [\underline{\nu}, \phi_1, \phi_1^*, \phi_2, \phi_2^*, \cdots, \phi_n, \phi_n^*]^T. \tag{39}$$

Therefore, we have

$$e^{At} = \Phi D_E \Phi^{-1} = \Phi D_E \Phi^T, \tag{40}$$

where

$$D_E = \begin{bmatrix} e^{0t} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{\beta_1 ti} & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & e^{-\beta_1 ti} & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & e^{\beta_2 ti} & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\beta_2 ti} & \vdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \vdots & e^{\beta_n ti} & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & e^{-\beta_n ti} \end{bmatrix}_{(2n+1)\times(2n+1)}. \tag{41}$$

If $\beta_1 = \beta_2 = \cdots = \beta_n = k$, and $t = \dfrac{\pi}{k}$, we have

$$e^{At} = e^{A\pi} = \Phi D_x \Phi^{-1} = \Phi\{D_1 + D_2\}\Phi^T = -I + 2\underline{\nu}\underline{\nu}^T = -H, \quad (42)$$

where k is a real constant and

$$D_x = D_1 + D_2, \quad (43)$$

in which

$$D_x = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \vdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \vdots & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & -1 \end{bmatrix}_{(2n+1)\times(2n+1)}, \quad (44)$$

$$D_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \vdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \vdots & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & -1 \end{bmatrix}_{(2n+1)\times(2n+1)}, \quad (45)$$

$$D_2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \end{bmatrix}_{(2n+1)\times(2n+1)} . \quad (46)$$

Therefore, $e^{A\pi}$ is found to be $-H$. The Householder matrix constructed by the $\underset{\sim}{\nu}$ vector of dimension $(2n+1)$ can be expressed by

$$e^{A\pi} = -H, \quad (47)$$

where A is a $(2n+1)$ by $(2n+1)$ anti-symmetric matrix with one zero eigenvalue and eigenvector $\underset{\sim}{\nu}$.

6. Relationship Between Real Symmetric Matrix B and e^{iBt}

Given a unit vector $\underset{\sim}{\nu}$, we can have a symmetric matrix B with a dimension of n by n, where $B = \underset{\sim}{\nu}\underset{\sim}{\nu}^T$. The corresponding eigenvalues of B are 1, and $n-1$ zeros. The corresponding eigenvector for the eigenvalue of 1 is $\underset{\sim}{\nu}$. Based on the Similarity Theorem for Matrix, we have

$$iB = \Psi D_B \Psi^{-1} = \Psi D_B \Psi^T, \quad (48)$$

where

$$\Psi = \left[\ \underset{\sim}{\nu} \quad \varphi_1 \quad \varphi_2 \quad \cdots \quad \varphi_{n-1}\ \right], \quad (49)$$

$$\Psi^{-1} = \Psi^T = \left[\ \underset{\sim}{\nu} \quad \varphi_1 \quad \varphi_2 \quad \cdots \quad \varphi_{n-1}\ \right]^T, \quad (50)$$

$$D_B = \begin{bmatrix} i & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \end{bmatrix}_{n\times n}, \tag{51}$$

$$D_y = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \vdots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 1 \end{bmatrix}_{n\times n}. \tag{52}$$

By setting the value of t to be π, we have

$$e^{iB\pi} = \Psi D_y \Psi^T = \Psi(D_3 - D_4)\Psi^T = I - 2\underset{\sim}{v}\bar{v}^T = H, \tag{53}$$

where

$$D_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \vdots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 1 \end{bmatrix}_{n\times n}, \tag{54}$$

$$D_4 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 \end{bmatrix}_{n\times n} . \tag{55}$$

The n by n even-order Householder matrix, H, constructed by using the $\underset{\sim}{v}$ vector can be expressed as

$$e^{iB\pi} = H , \tag{56}$$

where B is a symmetric matrix with eigenvalues 1 and 0. The corresponding eigenvector of the eigenvalue 1 is $\underset{\sim}{v}$.

7. Unique and Nonunique Solutions for A to Satisfy $e^{A\pi} = H$

In this section, we illustrate by using anti-symmetric matrices with dimensions of 5 by 5 and 3 by 3 to construct the Householder matrix using the matrix exponential.

Example 1. By setting $\nu_5 = \begin{Bmatrix} a \\ b \\ c \\ d \\ e \end{Bmatrix}$, we have

$$H_5 = I_5 - 2\frac{\nu_5 \nu_5^T}{\nu_5^T \nu_5} = \begin{bmatrix} 1-2a^2 & -2ab & -2ac & -2ad & -2ae \\ -2ba & 1-2b^2 & -2bc & -2bd & -2be \\ -2ca & -2cb & 1-2c^2 & -2cd & -2ce \\ -2da & -2db & -2dc & 1-2d^2 & -2de \\ -2ea & -2eb & -2ec & -2ed & 1-2e^2 \end{bmatrix}, \quad (57)$$

where $|\nu_5| = \nu_5^T \nu_5 = 1$. In order to satisfy $A\underset{\sim}{\nu} = 0$, we have many choices for A as shown below:

$$A_1 = \begin{bmatrix} 0 & -c & b & -e & d \\ c & 0 & \times & \times & \times \\ -b & \times & 0 & \times & \times \\ e & \times & \times & 0 & \times \\ -d & \times & \times & \times & 0 \end{bmatrix}, \quad (58)$$

$$A_2 = \begin{bmatrix} 0 & c & \times & \times & \times \\ -c & 0 & a & -e & d \\ \times & -a & 0 & \times & \times \\ \times & e & \times & 0 & \times \\ \times & -d & \times & \times & 0 \end{bmatrix}, \quad (59)$$

$$\vdots$$

where " × " denotes an arbitrary real number.

Example 2. By setting $\nu_3 = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$, we have

$$H_3 = I_3 - 2\frac{\nu_3 \nu_3^T}{\nu_3^T \nu_3} = \begin{bmatrix} 1-2a^2 & -2ab & -2ac \\ -2ba & 1-2b^2 & -2bc \\ -2ca & -2cb & 1-2c^2 \end{bmatrix}. \quad (60)$$

To satisfy $A\underset{\sim}{\nu} = 0$, we have only one choice for the anti-symmetric matrix,

$$A = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} . \tag{61}$$

Since the order of matrix A is n, the number of undetermined components of $\underset{\sim}{\nu}$ vector is $n-1$. The number of undetermined elements in matrix A is $\frac{n(n-1)}{2} - 1$. In the case of $n = 3$, we have only one solution since the number of unknowns are the same as the number of equations. For the case of $n > 3$, nonunique choices can be obtained since the number of unknowns is larger.

8. Numerical Examples

8.1. Numerical Examples of Real Anti-symmetric Matrices

1. Square matrix with a dimension of one by one

By setting $\underset{\sim}{\nu}_1 = 1$, we have $-e^{A_1 t} = 1 = H_1$, where $A_1 = [0]$, and $H_1 = [-1]$.

2. Square matrix with a dimension of three by three

By setting $\underset{\sim}{\nu}_3 = \begin{Bmatrix} 0.6 \\ 0 \\ 0.8 \end{Bmatrix}$, we have $-e^{A_3 \pi} = H_3$,

where $A_3 = \begin{bmatrix} 0 & -0.8 & 0 \\ 0.8 & 0 & -0.6 \\ 0 & 0.6 & 0 \end{bmatrix}$, and $H_3 = \begin{bmatrix} 0.28 & 0 & -0.96 \\ 0 & 1 & 0 \\ -0.96 & 0 & -0.28 \end{bmatrix}$.

3. Square matrix with a dimension of five by five

By setting $\underset{\sim}{\nu}_5 = \begin{Bmatrix} 0.1 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.7 \end{Bmatrix}$, we have $-e^{A_5 t} = H_5$, where $H_5 = \begin{bmatrix} 0.98 & -0.06 & -0.08 & -0.1 & -0.14 \\ -0.06 & 0.82 & -0.24 & -0.3 & -0.42 \\ -0.08 & -0.24 & 0.68 & -0.4 & -0.56 \\ -0.1 & -0.3 & -0.4 & 0.5 & -0.7 \\ -0.14 & -0.42 & -0.56 & -0.7 & 0.02 \end{bmatrix}$, and there are many other choices for A_5 as shown below:

$$A_{5(1)} = \begin{bmatrix} 0 & -0.4 & 0.3 & -0.7 & 0.5 \\ 0.4 & 0 & -0.772727 & -0.00909091 & 0.390909 \\ -0.3 & 0.772727 & 0 & -0.390909 & -0.00909091 \\ 0.7 & 0.00909091 & 0.390909 & 0 & -0.327273 \\ -0.5 & -0.390909 & 0.00909091 & 0.00909091 & 0.327273 \end{bmatrix}, \quad (62)$$

$$A_{5(2)} = \begin{bmatrix} 0 & -0.4 & 0.869231 & -0.1 & -0.253846 \\ 0.4 & 0 & -0.1 & 0.7 & -0.5 \\ -0.869231 & 0.1 & 0 & 0.253846 & -0.1 \\ 0.1 & -0.7 & 0.253846 & 0 & 0.430769 \\ 0.253846 & 0.5 & 0.1 & -0.430769 & 0 \end{bmatrix}, \quad (63)$$

$$A_{5(3)} = \begin{bmatrix} 0 & -0.928571 & 0.3 & 0.0571429 & 0.185714 \\ 0.928571 & 0 & -0.1 & -0.185714 & 0.0571429 \\ -0.3 & 0.1 & 0 & -0.7 & 0.5 \\ -0.0571429 & 0.185714 & 0.7 & 0 & -0.471429 \\ -0.185714 & -0.0571429 & -0.5 & 0.471429 & 0 \end{bmatrix}. \quad (64)$$

All the above matrices are Householder types. For the higher order cases, more choices can be obtained.

8.2. Numerical Examples of Real Symmetric Matrices

1. Square matrix with a dimension of one by one

By setting $\underline{\nu}_1 = 1$, we have $e^{iB_1t} = H_1$, where $H_1 = -1$, and $B_1 = [1]$.

2. Square matrix with a dimension of two by two

By setting $\underline{\nu}_2 = \left\{ \begin{array}{c} 0.6 \\ 0.8 \end{array} \right\}$, we have $e^{iB_2t} = H_2$, where

$$H_2 = \begin{bmatrix} 0.28 & -0.96 \\ -0.96 & -0.28 \end{bmatrix}, \quad \text{and} \quad B_2 = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}.$$

3. Square matrix with a dimension of three by three

By setting $\underline{\nu}_3 = \left\{ \begin{array}{c} 0.6 \\ 0 \\ 0.8 \end{array} \right\}$, we have $e^{iB_3t} = H_3$, where

$$H_3 = \begin{bmatrix} 0.28 & 0 & -0.96 \\ 0 & 1 & 0 \\ -0.96 & 0 & -0.28 \end{bmatrix}, \quad \text{and} \quad B_3 = \begin{bmatrix} 0.36 & 0 & 0.48 \\ 0 & 0 & 0 \\ 0.48 & 0 & 0.64 \end{bmatrix}.$$

4. Square matrix with a dimension of four by four

By setting $\underline{\nu}_4 = \left\{ \begin{array}{c} 0.1 \\ 0.5 \\ 0.5 \\ 0.7 \end{array} \right\}$, we have $e^{iB_4t} = H_4$, where

$$H_4 = \begin{bmatrix} 0.98 & -0.1 & -0.1 & -0.14 \\ -0.1 & 0.5 & -0.5 & 0.7 \\ -0.1 & -0.5 & 0.5 & -0.7 \\ -0.14 & -0.7 & 0.7 & 0.02 \end{bmatrix},$$

and

$$B_4 = \begin{bmatrix} 0.01 & 0.05 & 0.05 & 0.07 \\ 0.05 & 0.25 & 0.25 & 0.35 \\ 0.05 & 0.25 & 0.25 & 0.35 \\ 0.07 & 0.35 & 0.35 & 0.49 \end{bmatrix}.$$

5. Square matrix with a dimension of five by five

By setting $\underline{v}_5 = \begin{Bmatrix} 0.1 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.7 \end{Bmatrix}$, we have $e^{iB_5t} = H_5$, where

$$H_5 = \begin{bmatrix} 0.98 & -0.06 & -0.08 & -0.1 & -0.14 \\ -0.06 & 0.82 & -0.24 & -0.3 & -0.42 \\ -0.08 & -0.24 & 0.68 & -0.4 & -0.56 \\ -0.1 & -0.3 & -0.4 & 0.5 & -0.7 \\ -0.14 & -0.42 & -0.56 & -0.7 & 0.02 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 0.01 & 0.03 & 0.04 & 0.05 & 0.07 \\ 0.03 & 0.09 & 0.12 & 0.15 & 0.21 \\ 0.04 & 0.12 & 0.16 & 0.2 & 0.28 \\ 0.05 & 0.15 & 0.2 & 0.25 & 0.35 \\ 0.07 & 0.21 & 0.28 & 0.35 & 0.49 \end{bmatrix}.$$

9. Conclusions

In the paper, we have constructed the Householder matrix by using the matrix exponentials. Also, their relations between matrix exponentials and Householder matrices were discussed. Two cases for A in e^{At} were considered. One is an anti-symmetric matrix exponential with odd dimension, the other is a symmetric matrix exponential with even dimension. It is interesting to find that $e^{(B+iA)\pi}$ is also an orthogonal matrix, where the matrix $B + iA$ is a Hermitian matrix. Several examples have been shown to check the validity of the expressions. The inclusive and exclusive sets for the orthogonal matrices are summarized in Figure 2.

Figures

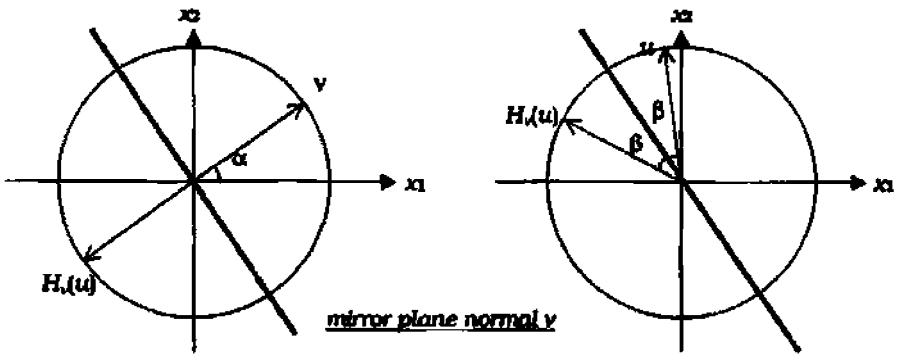

Figure 1: Mirror plane and mirror-mapped vector

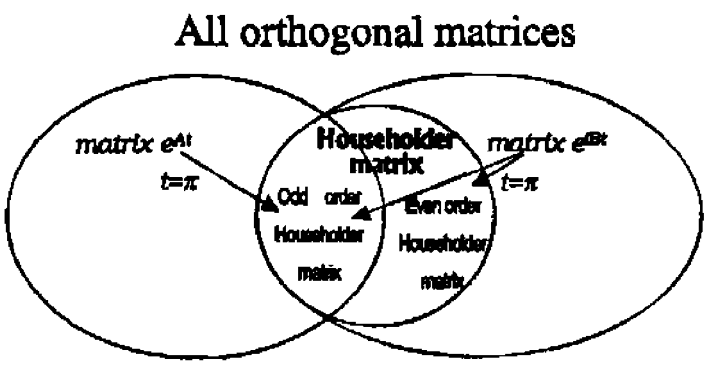

Figure 2: Relationship between the orthogonal matrices and the Householder matrices

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