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RIGID LINE INCLUSIONS UNDER ANTI-PLANE DEFORMATION AND IN-PLANE ELECTRIC FIELD IN PIEZOELECTRIC MATERIALS

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Abstract—The problems of the rigid line inclusions under anti-plane deformation and in-plane electric field in piezoelectric materials are studied by using the complex variable method of Muskhelishvili. The rigid line inclusions are considered, respectively, as a dielectric and conductor. A square-root singularity of the field variables is identified. The singularity exists only if the shear loading is acting perpendicular to the rigid lines and the electric loading is parallel to them. For the rigid dielectric line inclusion, the singularity coefficients of both stress and electric field depend on the material properties, and the mechanical and electrical loads applied at infinity, whereas the electric displacement is a constant on the entire plane. The values of the J-integral are always negative. For the rigid conductor line inclusion, the field singularity coefficients corresponding to the field variables applied at infinity become independent of the material properties and other respective loading parameters. The values of the J-integral can be either positive or negative. Copyright © 1997 Elsevier Science Ltd

1. INTRODUCTION

IT IS known that piezoelectric materials are widely used in technology. When subjected to mechanical and electrical loads, these piezoelectric materials can fail prematurely due to defects produced during their manufacturing process. It is therefore important to know how the defects disturb the field variables and how the stress concentration arises due to the existence of the defects. Crack problems in the piezoelectric materials have recently received much attention and have been investigated by many experts. The crack problems of mode III was studied in ref. [1] in which it was found that the crack growth can be either enhanced or retarded depending on the magnitude, the direction, and the type of the applied mechanical and electrical loads. Especially for certain ratios of the applied electrical load to mechanical load, crack arrestment can be observed. These findings are very important for engineering. Some other works about a crack or cavity can be found in refs [2–4]. In addition, Hao and Shen [5] proposed a new electric boundary condition from which the permittivity of air is considered when an electric field crosses the crack gap. More recently, based on the work of Dunn and Taya [6], the electroelastic field concentrations around ellipsoidal inhomogeneities in piezoelectric solids have been discussed [7]. However, they are far from satisfying engineering in practice. When a flat inclusion is much "harder" than the matrix, the rigid line model has to be considered in two-dimensional or longitudinal shear problems. Therefore, it is very important for us to study the rigid line problems (sometimes called hard crack or inverse crack problems) although these problems have been extensively studied in the case of purely elastic bodies.

The main objective of this work is to provide the basic characteristics of both the J-integral and field variables near the tip of a rigid line inclusion. It also provides a better understanding when the rigid line inclusions may be considered, respectively, as a dielectric and conductor.

2. STATEMENT OF PROBLEM

We consider the anti-plane strain deformation and the in-plane electric field in which some rigid flat inhomogeneities are embedded in an infinite body along the x-axis of a cartesian coordinate system xyz (Fig. 1). The deformation and electric field simplified can be obtained for the transversely isotropic piezoelectric materials [8] so long as some field variables vanish.

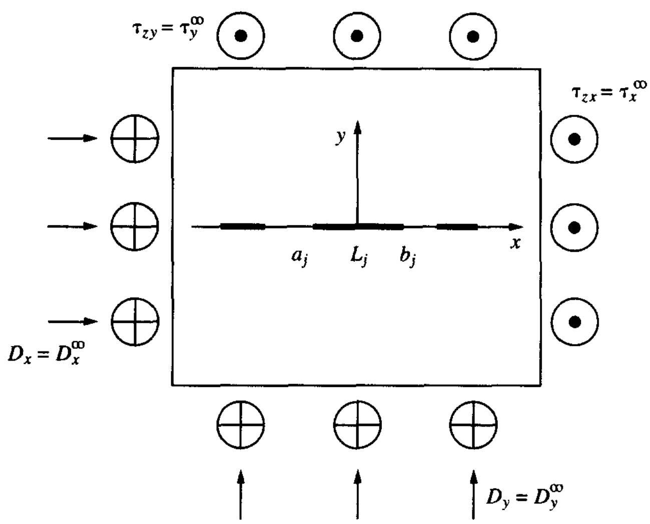

Fig. 1. Collinear rigid line inclusions, and far-field mechanical and electrical loads.

According to the work of Pak [1], the constitutive relations and the governing equations can be written as

$$\tau_{zx} = c_{44}\frac{\partial u_z}{\partial x} + e_{15}\frac{\partial \varphi}{\partial x}$$

$$\tau_{zy} = c_{44}\frac{\partial u_z}{\partial y} + e_{15}\frac{\partial \varphi}{\partial y}$$

$$D_x = e_{15}\frac{\partial u_z}{\partial x} - \epsilon_{11}\frac{\partial \varphi}{\partial x} \tag{1}$$

$$D_y = e_{15}\frac{\partial u_z}{\partial y} - \epsilon_{11}\frac{\partial \varphi}{\partial y}$$

$$\nabla^2[c_{44}u_z(x,y) + e_{15}\varphi(x,y)] = 0$$

$$\nabla^2[e_{15}u_z(x,y) - e_{11}\varphi(x,y)] = 0\,, \tag{2}$$

where τ_{zx} and τ_{zy} are the stresses, D_x and D_y are the electric displacement components, φ is the electric potential, u_z is the displacement, and c_{44} the elastic modulus, e_{15} the piezoelectric constant and ϵ_{11} the dielectric constant.

Because ∇^2 in eq. (2) is the two-dimensional Laplacian operator, u_z and φ can be the imaginary part of some analytic functions such that

$$u_z = \mathrm{Im}w(z) \tag{3}$$

$$\varphi = \mathrm{Im}\psi(z)\,. \tag{4}$$

So, the constitutive relations of eq. (1) can be written in the complex forms:

$$\tau_{zy} + \mathrm{i}\tau_{zx} = c_{44}W(z) + e_{15}\Psi(z) \tag{5}$$

$$D_y + \mathrm{i}D_x = e_{15}W(z) - \epsilon_{11}\Psi(z)\,, \tag{6}$$

where $W(z) = w'(z)$, $\Psi(z) = \psi'(z)$ and $z = x + \mathrm{i}y$. Because both the stress and electric displacement are limited at infinity as indicated in Fig. 1, there exist the forms:

$$W(z) = \Gamma_1 + W_0(z) \tag{7}$$

$$\Psi(z) = \Gamma_2 + \Psi_0(z)\,, \tag{8}$$

where $W_0(z) = \Psi_0(z) = 0$ at infinity, Γ_1 and Γ_2 are two constants which can be determined by substituting eqs (7) and (8) into eqs (5) and (6). At infinity, after simplifications, one has

$$\Gamma_1 = \frac{1}{c_{44}\epsilon_{11} + e_{15}^2}[(\epsilon_{11}\tau_y^\infty + e_{15}D_y^\infty) + \mathrm{i}(\epsilon_{11}\tau_x^\infty + e_{15}D_x^\infty)] \tag{9}$$

$$\Gamma_2 = \frac{1}{c_{44}\epsilon_{11} + e_{15}^2}[(e_{15}\tau_y^\infty - c_{44}D_y^\infty) + \mathrm{i}(e_{15}\tau_x^\infty - c_{44}D_x^\infty)]\,. \tag{10}$$

For the reason of convenience, we define new analytic functions as follows

$$V(z) = \bar{W}(z) = \overline{W(\bar{z})} = \bar{\Gamma}_1 + V_0(z) \tag{11}$$

$$\Phi(z) = \Psi(z) = \overline{\Psi(\bar{z})} = \bar{\Gamma}_2 + \Phi_0(z)\,, \tag{12}$$

where, clearly, $V_0(z) = \Phi_0(z) = 0$ at infinity.

As shown in Fig. 1, it is assumed that there is not an external load acting on the inclusions. Therefore, the mechanical boundary condition and the equilibrium condition of the inclusions are

$$u_z(t)^+ = u_z(t)^- = u_{oj} \qquad \text{on } L_j\ (j = 1{,}2\ldots\ldots n) \tag{13}$$

$$\int_{L_j}(\tau_{zy}^+ - \tau_{zy}^-)\mathrm{d}x = 0 \qquad \text{on } L_j\ (j = 1{,}2\ldots\ldots n)\,, \tag{14}$$

where the superscripts + and − refer, respectively, to the upper and lower inclusion surfaces, u_{oj} is constant. It follows from eq. (13) that

$$u_z'(t)^+ = 0,\ u_z'(t)^- = 0 \qquad \text{on } L \tag{15}$$

where

$$u_z'(t) = \frac{\mathrm{d}}{\mathrm{d}t}u_z(t) \text{ and } L = \sum_j L_j.$$

Now, let us consider the electric boundary conditions. First, assume that the rigid line inclusions are a dielectric. It is well known that the continuity conditions across the interface without the surface charge density between two dielectrics can be written as

$$\mathbf{n}\cdot[\mathbf{D}] = 0,\ [\varphi] = 0\,, \tag{16}$$

where square bracket signifies a physical value jump across the interface and **n** is the unit normal vector perpendicular to the interface. Because the thickness of the inclusions has been assumed to be zero, the conditions

$$D_y(t)^+ = D_y(t)^- \qquad \text{on } L \tag{17}$$

$$\varphi(t)^+ = \varphi(t)^- \qquad \text{on } L \tag{18}$$

would prevail on the inclusions.

Secondly, assume that the rigid line inclusions are a conductor. Based on the theory of electromagnetics which can be found from any book in this field, the electric potential is a constant around the conductor surface and the sum of the charges of electrostatic induction around the surface of each inclusion must be zero provided that the inclusions are not electrified. So, the electric boundary conditions for this case are the following

$$\varphi(t)^+ = \varphi(t)^- = \varphi_{oj} \qquad \text{on } L_j\ (j = 1{,}2\ldots\ldots n) \tag{19}$$

$$\int_{L_j}(D_y^+ - D_y^-)\mathrm{d}x = 0 \qquad \text{on } L_j\ (j = 1{,}2\ldots\ldots n)\,, \tag{20}$$

where φ_{oj} is constant. It follows from eq. (19) that

$$\varphi'(t)^{+} = 0,\ \varphi'(t)^{-} = 0 \qquad \text{on } L\,. \tag{21}$$

Hence, the problems have been reduced to find two analytic functions $W(z)$ and $\Psi(z)$ which must satisfy the boundary conditions (14), (15) either with eqs (17) and (18) or eqs (20) and (21) for respective cases of the dielectric inclusions and conductor inclusions.

3. ANALYSIS

Considering condition (15), it follows from eqs (3) and (11) and $W(z) = w'(z)$ that

$$W(t)^{+} - V(t)^{-} = 0,\ W(t)^{-} - V(t)^{+} = 0 \qquad \text{on } L\,, \tag{22}$$

which leads to the following equations

$$[W(t) + V(t)]^{+} - [W(t) + V(t)]^{-} = 0 \qquad \text{on } L \tag{23}$$

$$[W(t) - V(t)]^{+} + [W(t) - V(t)]^{-} = 0 \qquad \text{on } L\,. \tag{24}$$

From eq. (23), according to the Liouville theorem, the function $W(z) + V(z)$ is a constant on the entire plane. Considering eqs (7) and (11), one obtains

$$W(z) + V(z) = \Gamma_1 + \bar{\Gamma}_1\,. \tag{25}$$

Based on the theory of Muskhelishvili [9], the general solution to the Riemann–Hilbert problem [eq. (24)] is the following

$$W(z) - V(z) = P(z)X_0(z)\,, \tag{26}$$

where $P(z)$ is a polynomial

$$P(z) = C_n z^n + C_{n-1}z^{n-1} + \cdots + C_0 \tag{27}$$

and

$$X_0(z) = \prod_{j=1}^{n} = (z - a_j)^{-\frac{1}{2}}(z - b_j)^{-\frac{1}{2}} \tag{28}$$

and the single value branch will be determined when the condition $\lim_{|z|\to\infty} z^n X_0(z) = 1$ is satisfied. From eqs (25) and (26), one obtains

$$W(z) = \frac{1}{2}P(z)X_0(z) + \frac{1}{2}(\bar{\Gamma}_1 + \bar{\Gamma}_1) \tag{29}$$

$$V(z) = \frac{-1}{2}P(z)X_0(z) + \frac{1}{2}(\bar{\Gamma}_1 + \bar{\Gamma}_1)\,. \tag{30}$$

It follows from eqs (7), (9), (11) and (26) that

$$C_n = \Gamma_1 - \bar{\Gamma}_1 = \frac{2\mathrm{i}}{c_{44}\epsilon_{11} + e_{15}^2}(\epsilon_{11}\tau_x^{\infty} + e_{15}D_x^{\infty})\,. \tag{31}$$

Inserting the real part of eq. (5) into eq. (14), with the use of eqs (11) and (12), one has

$$\int_{L_j}\{[c_{44}(W - V) + e_{15}(\Psi - \Phi)]^{+} - [c_{44}(W - V) + e_{15}(\Psi - \Phi)]^{-}\}\mathrm{d}t = 0\,, \tag{32}$$

which can be put into the closed path integral

$$\oint_{\cap_j}\{c_{44}[W(z) - V(z)] + e_{15}[\Psi(z) - \Phi(z)]\}\mathrm{d}z = 0 \qquad j = 1,2\cdots,n \tag{32a}$$

where $\cap_j$ represents the closed path around each inclusion. Clearly, the constants C_{n-1}, C_{n-2},..., C_0 can be obtained when the functions $\Psi(z)$ and $\Phi(z)$ are known.

3.1. *The rigid dielectric line inclusions*

For the case of the rigid dielectric line inclusions, boundary conditions (17) and (18) will be used. Applying the real part of eq. (6) to eq. (17), with the aid of eqs (11) and (12), yields

$$[e_{15}(W - V) - \epsilon_{11}(\Psi - \Phi)]^+ - [e_{15}(W - V) - \epsilon_{11}(\Psi - \Phi)]^- = 0 \qquad \text{on } L . \tag{33}$$

According to the Liouville theorem, the function $e_{15}[W(z) - V(z)] - \epsilon_{11}[\Psi(z) - \Phi(z)]$ is a constant on the entire plane. It follows from eqs (7) – (12) that

$$e_{15}[W(z) - V(z)] - \epsilon_{11}[\Psi(z) - \Phi(z)] = e_{15}(\Gamma_1 - \bar{\Gamma}_1) - \epsilon_{11}(\Gamma_2 - \bar{\Gamma}_2) = 2iD_x^\infty . \tag{34}$$

Introducing eq. (4) into eq. (18) with the derivation with respect to t gives

$$\psi'(t)^+ - \overline{\psi'(t)^+} = \psi'(t)^- - \overline{\psi'(t)^-} \text{ on } L,$$

which leads to

$$[\Psi(t) + \Phi(t)]^+ - [\Psi(t) + \Phi(t)]^- = 0 , \tag{35}$$

where relation (12) and $\psi'(z) = \Psi(z)$ have been used. Hence, according to the Liouville theorem, the function $\Psi(z) + \Phi(z)$ is a constant on the entire plane. Recalling eqs (8), (10) and (12), one has

$$\Psi(z) + \Phi(z) = \Gamma_2 + \bar{\Gamma}_2 = \frac{2}{c_{44}\epsilon_{11} + e_{15}^2}(e_{15}\tau_y^\infty - c_{44}D_y^\infty) . \tag{36}$$

It follows from eqs (34) and (36) that

$$\begin{aligned}\Psi(z) &= \frac{e_{15}}{2\epsilon_{11}}[W(z) - V(z)] + \Gamma_2 - \frac{e_{15}}{2\epsilon_{11}}(\Gamma_1 - \bar{\Gamma}_1) \\ &= \frac{e_{15}}{2\epsilon_{11}}[W(z) - V(z)] + \frac{e_{15}\tau_y^\infty - c_{44}D_y^\infty}{c_{44}\epsilon_{11} + e_{15}^2} - \frac{iD_x^\infty}{\epsilon_{11}} .\end{aligned} \tag{37}$$

Substituting eq. (34) into eq. (32), we obtain

$$\oint_{\cap_j}\left\{\frac{c_{44}\epsilon_{11} + e_{15}^2}{\epsilon_{11}}[W(z) - V(z)] - \frac{2ie_{15}}{\epsilon_{11}}D_x^\infty\right\}dz = 0.$$

Inserting eq. (26) in the above integrals and neglecting some constants which have no effect on our problem gives

$$\oint_{\cap_j}P(z)X_0(z)dz = \oint_{\cap_j}[C_nz^n + C_{n-1}z^{n-1} + \cdots + C_0]X_0(z)dz = 0 . \tag{38}$$

Clearly, the closed path integrals will give the values of constants C_{n-1}, C_{n-2},...C_0.

3.2. *The rigid conductor line inclusions*

For the case of the rigid conductor line inclusions, boundary conditions (20) and (21) will be used. Substituting eq. (4) into eq. (21), it follows from $\psi'(z) = \Psi(z)$ and eq. (12) that

$$[\Psi(t) + \Phi(t)]^+ - [\Psi(t) + \Phi(t)]^- = 0 \qquad \text{on } L \tag{39}$$

$$[\Psi(t) - \Phi(t)]^+ + [\Psi(t) - \Phi(t)]^- = 0 \qquad \text{on } L . \tag{40}$$

According to the Liouville theorem, it is known from eq. (39) that the function $\Psi(z) + \Psi(z)$ is a constant on the entire plane. Considering eqs (8), (10) and (12), one has

$$\Psi(z) + \Phi(z) = \Gamma_2 + \bar{\Gamma}_2 = \frac{2}{c_{44}\epsilon_{11} + e_{15}^2}(e_{15}\tau_y^\infty - c_{44}D_y^\infty) . \tag{41}$$

Based on the theory of Muskhelishvili [9] the general solution to the Riemann–Hilbert problem [eq. (40)] is the following

$$\Psi(z) - \Phi(z) = Q(z)X_0(z)\,, \tag{42}$$

where $Q(z)$ is a polynomial

$$Q(z) = D_n z^n + D_{n-1}z^{n-1} + \cdots + D_0\,. \tag{43}$$

From eqs (41) and (42), one has

$$\Psi(z) = \frac{1}{2}Q(z)X_0(z) + \frac{1}{2}(\Gamma_2 + \bar{\Gamma}_2)\,. \tag{44}$$

Moreover, recalling eqs (8), (10) and (12), we have

$$D_n = \Gamma_2 - \bar{\Gamma}_2 = \frac{2\mathrm{i}}{c_{44}\epsilon_{11} + e_{15}^2}(e_{15}\tau_x^\infty - c_{44}D_x^\infty)\,. \tag{45}$$

The unknown constants remaining in eqs (27) and (43) can be determined by using eqs (20) and (32). Substituting the real part of eq. (6) into eq. (20) and using eqs (11) and (12), one has

$$\int_{L_j}\{[e_{15}(W - V) - \epsilon_{11}(\Psi - \Phi)]^+ - [e_{15}(W - V) - \epsilon_{11}(\Psi - \Phi)]^-\}\mathrm{d}t = 0 \qquad j = 1,2,\cdots,n$$

which are equivalent to the following closed path integrals

$$\oint_{\Omega_j}\{e_{15}[W(z) - V(z)] - \epsilon_{11}[\Psi(z) - \Phi(z)]\}\mathrm{d}z = 0 \qquad j = 1.2,\cdots,n\,. \tag{46}$$

Hence, applying eqs (26) and (42) to eqs (32) and (46), we arrive at

$$\oint_{\Omega_j}[(c_{44}C_n + e_{15}D_n)z^n + (c_{44}C_{n-1} + e_{15}D_{n-1})z^{n-1} + \cdots + (c_{44}C_0 + e_{15}D_0)]X_0(z)\mathrm{d}z = 0 \qquad j = 1,2,\cdots n\,. \tag{47}$$

$$\oint_{\Omega_j}[(e_{15}C_n - \epsilon_{11}D_n)z^n + (e_{15}C_{n-1} - \epsilon_{11}D_{n-1})z^{n-1} + \cdots + (e_{15}C_0 - \epsilon_{11}D_0)]X_0(z)\mathrm{d}z = 0 \qquad j = 1,2,\cdots n\,. \tag{48}$$

4. SINGULARITY OF FIELD VARIABLES

In the previous section, the closed form solutions have been reduced to find some constants which can be determined by using the closed path integrals. In analysis, the boundary conditions of mechanical and electrical loads at infinity are given by stresses and electrical displacement components which can be transformed equivalently to strains and electric field components through the constitutive relations (1)

$$\gamma_x^\infty = \frac{\epsilon_{11}\tau_x^\infty + e_{15}D_x^\infty}{c_{44}\epsilon_{11} + e_{15}^2}\,,\; E_x^\infty = \frac{c_{44}D_x^\infty - e_{15}\tau_x^\infty}{c_{44}\epsilon_{11} + e_{15}^2} \tag{49}$$

$$\gamma_y^\infty = \frac{\epsilon_{11}\tau_y^\infty + e_{15}D_y^\infty}{c_{44}\epsilon_{11} + e_{15}^2}\,,\; E_y^\infty = \frac{c_{44}D_y^\infty - e_{15}\tau_y^\infty}{c_{44}\epsilon_{11} + e_{15}^2}\,.$$

Considering $W(z) = w'(z)$ and $\Psi(z) = \psi'(z)$, it follows from eqs (3) and (4) that

$$\gamma_{zy} + \mathrm{i}\gamma_{zx} = \frac{\partial u_z}{\partial y} + \mathrm{i}\frac{\partial u_x}{\partial x} = W(z) \tag{50}$$

$$E_y + \mathrm{i}E_x = -\frac{\partial\phi}{\partial y} - \mathrm{i}\frac{\partial\phi}{\partial x} = -\Psi(z)\,. \tag{51}$$

Now, the field variables will be particularly discussed for a single rigid line as shown in Fig. 2.

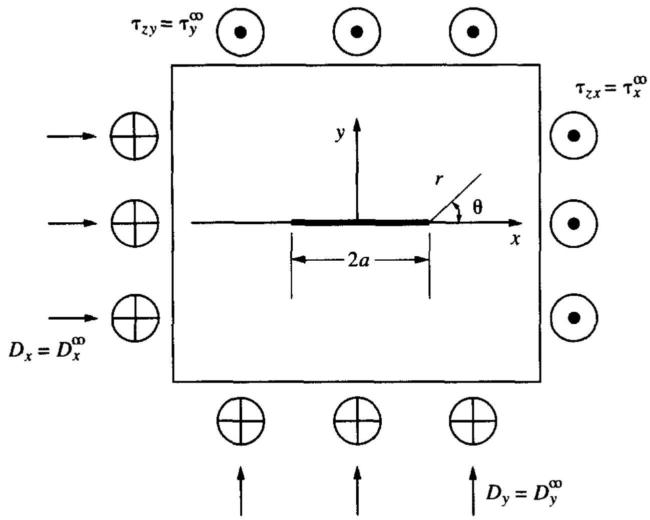

Fig. 2. Rigid dielectric line inclusion or rigid conductor line inclusion.

4.1. *The rigid dielectric line inclusion*

In this case, one has

$$P(z) = C_1 z + C_0,\ C_1 = \Gamma_1 - \bar{\Gamma}_1.$$

Substituting eq. (52) in the closed path integral (38) yields $C_0 = 0$. From the formulae given previously, all field variables can be written as

$$\tau_{zy} + \mathrm{i}\tau_{zx} = \frac{(c_{44}\epsilon_{11} + e_{15}^2)(\Gamma_1 - \bar{\Gamma}_1)}{2\epsilon_{11}} \cdot \frac{z}{\sqrt{z^2 - a^2}} + \text{const}$$

$$= \mathrm{i}\left(\tau_x^\infty + \frac{e_{15}}{\epsilon_{11}} D_x^\infty\right)\frac{z}{\sqrt{z^2 - a^2}} + \text{const} \tag{52}$$

$$\gamma_{zy} + \mathrm{i}\gamma_{zx} = \frac{(\Gamma_1 - \bar{\Gamma}_1)}{2} \frac{z}{\sqrt{z^2 - a^2}} + \text{const} = \mathrm{i}\gamma_x^\infty \frac{z}{\sqrt{z^2 - a^2}} + \text{const} \tag{53}$$

$$D_y + \mathrm{i}D_x = D_y^\infty + \mathrm{i}D_x^\infty \tag{54}$$

$$E_y + \mathrm{i}E_x = -\frac{e_{15}(\Gamma_1 - \bar{\Gamma}_1)}{2\epsilon_{11}} \frac{z}{\sqrt{z^2 - a^2}} + \text{const}$$

$$= -\frac{\mathrm{i}e_{15}^2}{c_{44}\epsilon_{11}}\left(E_x^\infty + \frac{\tau_x^\infty}{e_{15}}\right)\frac{z}{\sqrt{z^2 - a^2}} + \text{const}\,. \tag{55}$$

Since we are interested in the field variables near the tip of the rigid line, it is convenient to choose a local polar coordinate system (r, θ) [Fig. 2] with the origin at a, i.e.

$$z = a + r\mathrm{e}^{\mathrm{i}\theta}\,. \tag{56}$$

With the aid of eq. (56), the functions (52)–(55) become

$$\tau_{zy} + \mathrm{i}\tau_{zx} = \frac{\mathrm{i}S^{\mathrm{T}}}{\sqrt{2\pi r}}\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}, \qquad \gamma_{zy} + \mathrm{i}\gamma_{zx} = \frac{\mathrm{i}S^{\mathrm{S}}}{\sqrt{2\pi r}}\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}$$

$$D_y + \mathrm{i}D_x = D_y^\infty + \mathrm{i}D_x^\infty, \qquad E_y + \mathrm{i}E_x = -\frac{\mathrm{i}S^{\mathrm{E}}}{\sqrt{2\pi r}}\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}, \tag{57}$$

where the higher terms are dropped and the field singularity coefficients are as follows

$$S^{\mathrm{T}} = (\tau_x^\infty + \frac{e_{15}}{\epsilon_{11}} D_x^\infty)\sqrt{\pi a},\ S^{\mathrm{S}} = \gamma_x^\infty \sqrt{\pi a} \tag{58}$$

$$S^{\mathrm{E}} = \frac{e_{15}^2}{c_{44}\epsilon_{11}}\left(E_x^\infty + \frac{\tau_x^\infty}{e_{15}}\right)\sqrt{\pi a}.$$

It is clear that apart from the electric displacement the system has a square-root singularity at the tip of the rigid line similar to the case of a slit crack [1]. However, unlike the cases of the crack problem [1] in piezoelectric material and the inverse crack problem [10] in purely elastic material, the stress singularity coefficient S^{T} and the electric field singularity coefficient S^{E} depend on both the material properties, and the mechanical and electrical loads. It is also worth noting that, in eq. (58), the applied mechanical load is acting perpendicular to the inclusion segment and the applied electric load is parallel to it.

The J-integral can be calculated from the work of Pak [1]

$$J = \int_s (Hn_x - T_i u_{i,x} + D_j n_j E_x)\mathrm{d}s\ , \tag{59}$$

where H is the electric enthalpy density and T_i is the applied surface traction. Using the solution obtained and evaluating this integral on a vanishingly small contour at the rigid line tip results in

$$J = -\frac{\pi a(\epsilon_{11}\tau_x^\infty + e_{15}D_x^\infty)^2}{2\epsilon_{11}(c_{44}\epsilon_{11} + e_{15}^2)}\ , \tag{60}$$

which shows that the values of the J-integral in this case are always negative. This characteristic is the same as that of the case of purely elastic materials. If the piezoelectric constant $e_{15} = 0$, the J-integral will coincide with the case of purely elastic materials [10]. Actually, the negative sign in eq. (60) indicates that the driving force on the rigid line favors a contraction in the length of the segment.

Considering relations in eq. (49), it can be easily found that if the equation

$$\epsilon_{11}\tau_x^\infty + e_{15}D_x^\infty = 0 \tag{61}$$

is satisfied, both the singularity of all field variables and the J-integral vanish. Examining eq. (60), one can use

$$J_e = +\frac{\pi a \tau_x^{\infty 2}}{2c_{44}} \tag{62}$$

as a normalization factor. Numerical values of the normalized J-integral J/J_e are plotted in Fig. 3 in which the material properties are given by [1]

$$c_{44} = 3.53 \times 10^{10}\left(\frac{N}{m^2}\right),\ e_{15} = 17.0\left(\frac{C}{m^2}\right),\ \epsilon_{11} = 151 \times 10^{-10}\left(\frac{C}{Vm}\right), \tag{63}$$

where N is the force in Newtons, m is the length in meters, C is the charge in Coulombs and V is the electric potential in volts.

4.2. *The rigid conductor line inclusion*

For this case, one can easily know

$$P(z) = C_1 z + C_0,\ Q(z) = D_1 z + D_0,\ C_1 = \Gamma_1 - \bar{\Gamma}_1,\ D_1 = \Gamma_2 - \bar{\Gamma}_2\,. \tag{64}$$

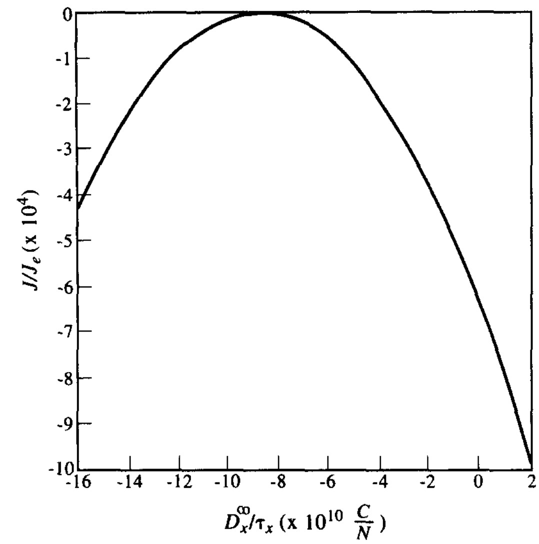

Fig. 3. Normalized J-integral of the rigid dielectric line inclusion with ratios of mechanical load to electrical load at infinity.

It follows, by using the closed path integrals (47) and (48), that $C_0 = D_0 = 0$. Using the formulae presented previously, we have

$$\tau_{zy} + \mathrm{i}\tau_{zx} = \mathrm{i}\tau_x^\infty \frac{z}{\sqrt{z^2 - a^2}} + \text{const}; \qquad \gamma_{zy} + \mathrm{i}\gamma_{zx} = \mathrm{i}\gamma_x^\infty \frac{z}{\sqrt{z^2 - a^2}} + \text{const}$$

$$D_y + \mathrm{i}D_x = \mathrm{i}D_x^\infty \frac{z}{\sqrt{z^2 - a^2}} + \text{const}; \qquad E_y + \mathrm{i}E_x = -\mathrm{i}E_x^\infty \frac{z}{\sqrt{z^2 - a^2}} + \text{const}. \tag{65}$$

Therefore, by using the local polar coordinate system as shown in Fig. 2, the field variables near the tip of the rigid line can be written as

$$\tau_{zy} + \mathrm{i}\tau_{zx} = \frac{\mathrm{i}S^{\mathrm{T}}}{\sqrt{2\pi r}} \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}, \qquad \gamma_{zy} + \mathrm{i}\gamma_{zx} = \frac{\mathrm{i}S^{\mathrm{S}}}{\sqrt{2\pi r}} \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}$$

$$D_y + \mathrm{i}D_x = \frac{\mathrm{i}S^{\mathrm{D}}}{\sqrt{2\pi r}} \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}, \qquad E_y + \mathrm{i}E_x = -\frac{\mathrm{i}S^{\mathrm{E}}}{\sqrt{2\pi r}} \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}, \tag{66}$$

where the field singularity coefficients are as follows

$$S^{\mathrm{T}} = \tau_x^\infty \sqrt{\pi a}, \; S^{\mathrm{S}} = \gamma_x^\infty \sqrt{\pi a}$$

$$S^{\mathrm{D}} = D_x^\infty \sqrt{\pi a}, \; S^{\mathrm{E}} = E_x^\infty \sqrt{\pi a}. \tag{67}$$

For this case, the field singularity coefficients corresponding to the field variables applied at infinity become independent of the material constants and the other respective loading parameter. Also, the applied mechanical load is acting perpendicular to the inclusion segment and the applied electric load is parallel to it.

The J-integral is calculated by inserting eq. (66) into eq. (59) on a vanishingly small contour at the rigid line tip

$$J = -\frac{\pi a(\epsilon_{11}\tau_x^{\infty 2} + 2e_{15}\tau_x^\infty D_x^\infty - c_{44}D_x^{\infty 2})}{2(c_{44}\epsilon_{11} + e_{15}^2)}. \tag{68}$$

Clearly, the values of this J-integral can be either positive or negative. Solving the roots of the quadratic eq. (68), the J-integral can be shown to be negative when

$$-\frac{\sqrt{c_{44}\epsilon_{11} + e_{15}^2} - e_{15}}{c_{44}} < \frac{D_x^\infty}{\tau_x^\infty} < \frac{\sqrt{c_{44}\epsilon_{11} + e_{15}^2} + e_{15}}{c_{44}}. \tag{69}$$

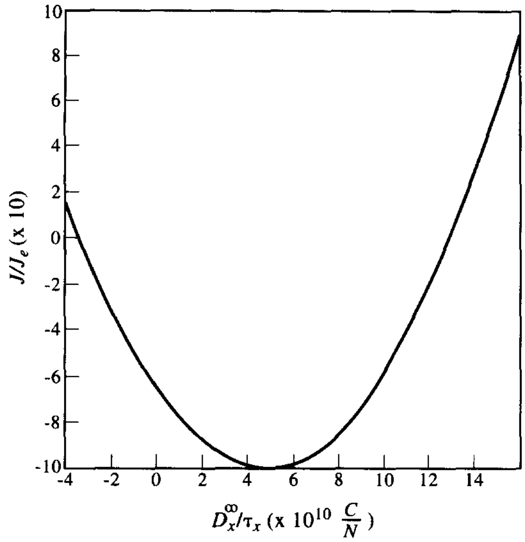

Fig. 4. Normalized J-integral of the rigid conductor line inclusion with ratios of mechanical load to electrical load at infinity.

Substituting the material properties of eq. (63) into eq. (69) gives

$$-3.30 \times 10^{-10} < D_x^\infty/\tau_x^\infty < 12.9 \times 10^{-10}(C/N)\,. \tag{70}$$

By using the normalization factor in eq. (62) and the material properties in eq. (63), the values of the normalized J-integral J/J_e are plotted in Fig. 4.

5. CONCLUSIONS

The collinear rigid line inclusions of either a dielectric or conductor have been considered under anti-plane deformation and in-plane electric field in piezoelectric materials. The closed form solutions are obtainable so long as we evaluate some closed path integrals to determine some constants. The characteristics of the field variables have been examined for a single rigid line. Comparing the rigid dielectric line inclusion with the rigid conductor line inclusion, it is found that the disturbed field variables are much more different. Physically, a conductor can be considered as a special dielectric for which the dielectric constant is infinitely great. Therefore, the model of the rigid conductor line inclusion is an extreme case for the rigid line inhomogeneity from the viewpoint of electrostatics.

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