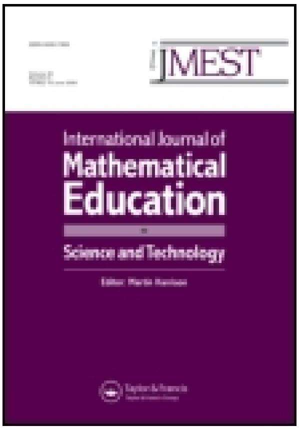

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Polar and singular value decomposition of 3×3 magic squares

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Polar and singular value decomposition of 3 × 3 magic squares

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In this note, we find polar as well as singular value decompositions of a 3×3 magic square, i.e. a 3×3 matrix $\mathbf{M}$ with real elements where each row, column and diagonal adds up to the magic sum s of the magic square.

Keywords: magic squares; polar decomposition; singular value decomposition; Luoshu

1. Introduction

Any 3×3 magic square can be represented as a 3×3 matrix $\mathbf{M}$ with real elements where each row, column and diagonal adds up to s, called its magic sum, i.e.

$$\mathbf{M1} = s\mathbf{1}\,, \quad \mathbf{1}'\mathbf{M} = s\mathbf{1}'\,, \quad \mathrm{tr}(\mathbf{M}) = s\,, \quad \mathrm{tr}(\mathbf{FM}) = s\,,$$

where $'$ denotes transposition, $\mathbf{1} = (1, 1, 1)'$, and left multiplication by the (flip) matrix $\mathbf{F} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ reverses the rows of $\mathbf{M}$ such that $\mathrm{tr}(\mathbf{FM})$ is equal to the sum of the elements of the anti-diagonal of $\mathbf{M}$.

According to Trenkler, Schmidt and Trenkler,[01] such a matrix can be written as

$$\mathbf{M} = s\mathbf{J} + \mathbf{N}\,,$$

where $\mathbf{J} = \frac{1}{3}\mathbf{11}'$ is an idempotent matrix, and

$$\mathbf{N} = \mathbf{N}(\alpha, \beta) = \begin{pmatrix} \alpha + \beta & -2\alpha & \alpha - \beta \\ -2\beta & 0 & 2\beta \\ -\alpha + \beta & 2\alpha & -\alpha - \beta \end{pmatrix}$$

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with real numbers α and β. Alternatively, the matrix $\mathbf{N}$, which is clearly magic with the magic sum zero, can be represented as

$$\mathbf{N} = \mathbf{N}(\alpha, \beta) = \alpha\mathbf{ab}' + \beta\mathbf{ba}' ,$$

where $\mathbf{a} = (1, 0, -1)'$ and $\mathbf{b} = (1, -2, 1)'$.

Although magic squares are fun, they also stimulate mathematical thinking. They present excellent examples for many significant areas of linear algebra such as rank, trace, inverses, eigenvalues and eigenvectors of matrices.

2. Polar decomposition

For the polar decomposition of a non-singular magic square $\mathbf{M}$, we wish to write $\mathbf{M} = \mathbf{PU}$, where $\mathbf{P}$ is a non-negative definite and $\mathbf{U}$ an orthogonal matrix. Since $\mathbf{M}$ is non-singular, $\mathbf{P}$ and $\mathbf{U}$ are unique and $\mathbf{P}$ is positive definite (see [2, Section 7.3]). Actually, we have $\mathbf{P} = (\mathbf{MM}')^{1/2}$, where $(\,\cdot\,)^{1/2}$ denotes the square root of a non-negative definite matrix.

It can be readily established that

$$\mathbf{MM}' = s^2\mathbf{J} + \mathbf{NN}' = s^2\mathbf{J} + 6\alpha^2\mathbf{aa}' + 2\beta^2\mathbf{bb}'$$

and

$$\mathbf{P} = (\mathbf{MM}')^{1/2} = \varphi\mathbf{J} + \psi\mathbf{aa}' + \varepsilon\mathbf{bb}' ,$$

where

$$\varphi = |s| , \quad \psi = \sqrt{3}|\alpha| , \quad \varepsilon = \frac{\sqrt{3}}{3}|\beta| .$$

According to Trenkler, Schmidt and Trenkler,[11] we have $\det(\mathbf{M}) = -12s\alpha\beta$ which implies that s, α and β have to be non-zero. Now we can derive the polar decomposition of $\mathbf{M}$ from the factorization $\mathbf{M} = \mathbf{PU}$ where the non-singular square root $\mathbf{P} = (\mathbf{MM}')^{1/2}$ is given above. Note that $\mathbf{P}$ is semi-magic since each row and column of $\mathbf{P}$ adds up to $|s|$.

It follows that $\mathbf{U} = \mathbf{P}^{-1}\mathbf{M}$, where

$$\mathbf{P}^{-1} = \frac{1}{\varphi}\mathbf{J} + \frac{1}{4\psi}\mathbf{aa}' + \frac{1}{36\varepsilon}\mathbf{bb}' .$$

Some further straightforward calculations give

$$\mathbf{U} = \frac{s}{|s|}\mathbf{J} + \frac{\sqrt{3}}{6}\left(\frac{\alpha}{|\alpha|}\mathbf{ab}' + \frac{\beta}{|\beta|}\mathbf{ba}'\right)$$

and

$$\mathbf{UU}' = \mathbf{J} + \frac{1}{2}\mathbf{aa}' + \frac{1}{6}\mathbf{bb}' = \mathbf{I} ,$$

the 3×3 identity matrix. Hence, in the polar decomposition $\mathbf{M} = \mathbf{PU}$ the matrix $\mathbf{U}$ is simultaneously orthogonal and magic with magic number $s/|s| = \text{sign}(s)$. Thus, the polar decomposition of a 3×3 magic square is the product of a semi-magic and a magic square.

3. Singular value decomposition

Using the representation of $\mathbf{MM}'$, we can also derive a singular value decomposition of $\mathbf{M}$, which in general is not unique (see [2, Section 7.3]). For this purpose consider the 3×3 matrix

$$\mathbf{S} = \left(\frac{\sqrt{3}}{3}\mathbf{1}, \frac{\sqrt{2}}{2}\mathbf{a}, \frac{\sqrt{6}}{6}\mathbf{b} \right).$$

It is clear that $\mathbf{S}$ is orthogonal and $\mathbf{MM}'\mathbf{S} = \mathbf{S\Lambda}$ with $\mathbf{\Lambda} = \text{diag}(s^2, 12\alpha^2, 12\beta^2)$, the diagonal matrix with diagonal elements s^2, $12\alpha^2$, $12\beta^2$. Having assumed that $\mathbf{M}$ is non-singular, we may define $\mathbf{\Lambda}^{-1/2} = (\mathbf{\Lambda}^{1/2})^{-1}$ with $\mathbf{\Lambda}^{1/2} = \text{diag}(|s|, 2\sqrt{3}|\alpha|, 2\sqrt{3}|\beta|)$. Then the matrix $\mathbf{T} = \mathbf{M}'\mathbf{S\Lambda}^{-1/2}$ is also orthogonal and $\mathbf{M}'\mathbf{MT} = \mathbf{T\Lambda}$. It follows that

$$\mathbf{M} = \mathbf{SS}'\mathbf{M} = \mathbf{S\Lambda}^{1/2}\mathbf{\Lambda}^{-1/2}\mathbf{S}'\mathbf{M} = \mathbf{S\Lambda}^{1/2}\mathbf{T}',$$

which is a singular value decomposition of $\mathbf{M}$. Note that $\mathbf{MM}'\mathbf{1} = s^2\mathbf{1}$, $\mathbf{MM}'\mathbf{a} = 12\alpha^2\mathbf{a}$ and $\mathbf{MM}'\mathbf{b} = 12\beta^2\mathbf{b}$, so that the singular values of $\mathbf{M}$ are indeed $|s|$, $2\sqrt{3}|\alpha|$ and $2\sqrt{3}|\beta|$.

We would also like to note that a 3×3 magic square $\mathbf{M} = s\mathbf{J} + \alpha\mathbf{ab}' + \beta\mathbf{ba}'$ is orthogonal if and only if $s^2 = 1$ and $\alpha^2 = 1/12 = \beta^2$. Setting $s = 1$ and $\alpha = 1/\sqrt{12} = \beta$, we get one of eight possible orthogonal magic squares, which is given by

$$\mathbf{M} = \frac{1}{3}\begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} & 1 \\ 1-\sqrt{3} & 1 & 1+\sqrt{3} \\ 1 & 1+\sqrt{3} & 1-\sqrt{3} \end{pmatrix}.$$

4. Example

As an example we apply these results to one of the most famous magic squares, the Luoshu, which dates back to ancient China and is given by

$$\mathbf{L} = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix},$$

see [3]. Since $\det(\mathbf{L}) = 360$, $\mathbf{L}$ is non-singular. Furthermore, we have $s = 15$, $\alpha = -2$ and $\beta = 1$, so that

$$\mathbf{L} = 15\mathbf{J} + \mathbf{N}(-2, 1) = 15\mathbf{J} - 2\mathbf{ab}' + \mathbf{ba}'.$$

The unique polar decomposition of the Luoshu is given by $\mathbf{L} = \mathbf{P_L U_L}$, where

$$\mathbf{P_L} = (\mathbf{LL}')^{1/2} = 15\mathbf{J} + 2\sqrt{3}\mathbf{aa}' + \frac{\sqrt{3}}{3}\mathbf{bb}' = 15\mathbf{J} + \frac{\sqrt{3}}{3}\begin{pmatrix} 7 & -2 & -5 \\ -2 & 4 & -2 \\ -5 & -2 & 7 \end{pmatrix}$$

and

$$\mathbf{U_L} = \mathbf{J} + \frac{\sqrt{3}}{6}(\mathbf{ba}' - \mathbf{ab}') = \mathbf{J} + \frac{\sqrt{3}}{3}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Furthermore, some straightforward calculations yield a singular value decomposition $\mathbf{L} = \mathbf{S_L \Lambda_L^{1/2} T_L'}$, where

$$\mathbf{S_L} = \frac{1}{6}\begin{pmatrix} 2\sqrt{3} & 3\sqrt{2} & \sqrt{6} \\ 2\sqrt{3} & 0 & -2\sqrt{6} \\ 2\sqrt{3} & -3\sqrt{2} & \sqrt{6} \end{pmatrix}, \qquad \mathbf{T_L} = \frac{1}{6}\begin{pmatrix} 2\sqrt{3} & -\sqrt{6} & 3\sqrt{2} \\ 2\sqrt{3} & 2\sqrt{6} & 0 \\ 2\sqrt{3} & -\sqrt{6} & -3\sqrt{2} \end{pmatrix}$$

and

$$\mathbf{\Lambda_L^{1/2}} = \begin{pmatrix} 15 & 0 & 0 \\ 0 & 4\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{3} \end{pmatrix}.$$

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