α and β are the yaw and pitch angles of the attitude of the reference frame M relative to F.

Now introduce the angles $\xi = \tilde{x}/R$ and $\eta = \tilde{y}/r$ as the spherical coordinates for **x** in M, defined in the same way as α and β, so we have:

$$\mathbf{x} = (R\sin\xi\cos\eta,\ R\sin\eta,\ R\cos\xi\cos\eta),$$

$$\text{or}\quad \mathbf{x} = (R\sin(\tilde{x}/R)\cos(\tilde{y}/R),\ R\sin(\tilde{y}/R),\ R\cos(\tilde{x}/R)\cos(\tilde{y}/r)), \tag{13}$$

$\tilde{x}$ and $\tilde{y}$ are arc-lengths measured along the surface of a sphere of radius R.

Substitute (12) and (13) into (2) and compute the series expansions of the sine and cosine functions to obtain

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & \tilde{a}/R \\ \sin\theta & \cos\theta & \tilde{b}/R \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \tilde{x} \\ \tilde{y} \\ R \end{Bmatrix} + (1/R)$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (-\tilde{a}\cos\theta + \tilde{b}\sin\theta) & (\tilde{a}\sin\theta + \tilde{b}\cos\theta) & -(\tilde{a}^2 + \tilde{b}^2)/R \end{bmatrix} \begin{Bmatrix} \tilde{x} \\ \tilde{y} \\ R \end{Bmatrix} + O(1/R^2). \tag{14}$$

The zeroth order term in (14) is the equation of a planar displacement in the $Z = R$ plane, parallel to the $X-Y$ coordinate plane of F. Also notice that the first order term consists only of a component in the Z direction. We conclude that the planar displacement (1) is the limiting case of the spherical displacement (2) for which the angles α, β, ξ and η are small such that the terms $\alpha^2 = (\tilde{a}/R)^2$, $\beta^2 = (\tilde{b}/R)^2$, $\xi^2 = (\tilde{x}/R)^2$ and $\eta^2 = (\tilde{y}/R)^2$ are negligible.

Blaschke's mapping is obtained as a limiting case of (11) by following the same procedure. Substitute $\alpha = \tilde{a}/R$ and $\beta = \tilde{b}/R$ into (11), compute the series expansion of the sine and cosine functions, and collect terms in $1/R$. The result is

$$X_1 = (\tilde{a}\sin(\theta/2) - \tilde{b}\cos(\theta/2))/2R + O(1/R^3),$$
$$X_2 = (\tilde{a}\cos(\theta/2) + \tilde{b}\sin(\theta/2))/2R + O(1/R^3),$$
$$X_3 = \sin(\theta/2) + O(1/R^2),$$
$$X_4 = \cos(\theta/2) + O(1/R^2). \tag{15}$$

These computations were facilitated using the symbolic computation software MACSYMA. The similarities between (15) and (6) are clear and we conclude that Blaschke's map is the limiting case of Ravani's for which the terms $\alpha^2 = (\tilde{a}/R)^2$ and $\beta^2 = (\tilde{b}/R)^2$ are negligible.

Conclusion

This note demonstrates that Blaschke's map of planar displacements is closely related to Ravani's map of spherical displacements which is based on Euler parameters. This result is of interest because it suggests a way to extend recent results on the differential geometry of the image space of spherical kinematics under Ravani's map (McCarthy and Ravani, 1985) to the image space of planar kinematics under Blaschke's map.

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Stress Singularity at the Tip of a Rigid Line Inhomogeneity Under Antiplane Shear Loading

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1 Introduction

It has been shown (Chou and Wang, 1983; Wang et al., 1985) that under an inplane tensile loading, a plane elastic body containing a rigid line inhomogeneity would generate a square-root stress singularity at the tip of the line segment, a situation similar to the case of a slit crack (Broek, 1982). However, if the applied stress is an inplane shear, the inhomogeneity and the matrix become elastically compatible and no stress singularity exists, which is different from the case of a crack.

In the present work we investigate a similar inhomogeneity system in which the applied stress is an antiplane shear. This system is analyzed by two different methods: the complex variable method of Muskhelishvili and the equivalent inclusion method of Eshelby. The present analysis, together with the analysis reported earlier for the inplane elastic system (Chou and Wang, 1983; Wang et al., 1985), thus provides a complete description for the rigid line inhomogeneity problem in plane elasticity.

2 Analysis

We consider an antiplane strain deformation in which a rigid flat inhomogeneity of width $2a$ is embedded in an infinite elastic body along the x axis of a Cartesian coordinate system xyz (Fig. 1). The dependent variables of the system are the displacement $u_z(x,y)$ and the stress components σ_{xz} and σ_{yz}, related by

$$\sigma_{xz} = G\frac{\partial u_z}{\partial x},\quad \sigma_{yz} = G\frac{\partial u_z}{\partial y} \tag{1}$$

where G is the shear modulus of the matrix. Under the condition of equilibrium, the governing partial differential equation is

$$\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} = 0 \tag{2}$$

with the boundary conditions

$$u_z = 0 \quad \text{for } -a \leqq x \leqq a \text{ and } y = 0 \tag{3}$$

and

$$\sigma_{xz} = \sigma_{xz}^A,\ \sigma_{yz} = \sigma_{yz}^A \quad \text{at infinity} \tag{4}$$

where σ_{xz}^A and σ_{yz}^A are the uniform applied stresses. Equation (2) can be solved according to the position of the loading.

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Manuscript received by ASME Applied Mechanics Division, August, 1985; final revision, October, 1985.

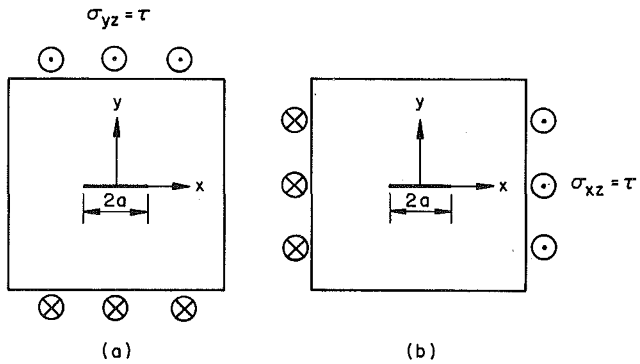

Fig. 1 A rigid line inhomogeneity under the action of an antiplane shear loading. (a) $\sigma_{xz}^A = o$, $\sigma_{yz}^A = \tau$, (b) $\sigma_{xz}^A = \tau$, $\sigma_{yz}^A = o$.

There are two distinct cases. In the first case (Fig. 1*a*) where the applied stress is acting parallel to the rigid inhomogeneity, $\sigma_{xz}^A = 0$ and $\sigma_{yz}^A = \tau$, and the solution of (2) is simply

$$u_z = \frac{\tau}{G} y \tag{5}$$

which satisfies both the boundary conditions (3) and (4). The corresponding stresses in the matrix are $\sigma_{xz} = 0$ and $\sigma_{yz} = \tau$. The inhomogeneity is, therefore, compatible with the matrix and no stress singularity exists.

In the second case (Fig. 1*b*) where the applied stress is acting perpendicular to the rigid inhomogeneity, $\sigma_{xz}^A = \tau$ and $\sigma_{yz}^A = 0$. A suggested solution of (2) is

$$u_z^s = \frac{\tau}{G} x \tag{6}$$

which satisfies the boundary condition (4) but not (3). For a valid solution, a perturbation field must be added with the following boundary conditions:

$$u_z^P = -\frac{\tau}{G} x \quad \text{for } -a \leqq x \leqq a \text{ and } y = 0 \tag{7}$$

and

$$\sigma_{xy}^P = \sigma_{yz}^P = 0 \quad \text{at infinity} \tag{8}$$

For solving the above boundary value problem, it is convenient to use the complex variable method of Muskhelishvili (1953). Consider the mapping function

$$z = \omega(\zeta) = \frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right) \tag{9}$$

which maps the rigid line in the z plane ($z = x + iy$) onto a unit circle in the ζ plane ($\zeta = \rho\, e^{i\psi}$). The entire z plane, excluding the rigid line, is then mapped onto the region exterior to the unit circle in the ζ plane. With the value of u_z^P given by (7) in the z plane, the corresponding value on the unit circle in the ζ plane can be determined. The problem is, therefore, a Dirichlet problem. The displacement function which is harmonic in the region exterior to the unit circle in the ζ plane can be represented by the Poisson's formula

$$\delta^P(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\rho^2 - 1)\delta^P(\phi)d\phi}{1 - 2\rho\cos(\phi - \psi) + \rho^2},\ \rho > 1 \tag{10}$$

where $\delta^P(\phi)$ is the value of $\delta^P(\zeta)$ on the unit circle $t = e^{i\phi}$ in the ζ plane. From equations (7) and (9), we have

$$\delta^P(\phi) = -\frac{\tau}{G} a \cos\phi \tag{11}$$

which is also the boundary condition for the displacement function in the ζ plane. Equation (10) can then be evaluated by the contour integration to yield

$$\delta^P(\zeta) = -\frac{\tau a}{G}\frac{1}{\rho}\cos\psi = -\frac{\tau a}{G}\frac{\mathrm{Re}\zeta}{|\zeta|^2} \tag{12}$$

where Reζ represents the real part of ζ. Equation (12) reduces to (11) at $\rho = 1$ and $\psi = \phi$, i.e., on the unit circle in the ζ plane.

To determine the displacement function in the z plane, we use the inverse function of (9)

$$\zeta = \omega^{-1}(z) = \frac{z + \sqrt{z^2 - a^2}}{a} \tag{13}$$

Since we are interested in the field near the tip of the rigid line, it is convenient to choose a local polar coordinate system (r, θ) with the origin at a, i.e.,

$$z = a + r\, e^{i\theta} \tag{14}$$

Substituting equation (14) in equation (13) and assuming $r \ll a$ yield

$$\zeta = 1 + \left(\frac{2r}{a}\right)^{1/2} e^{i\theta/2} + \frac{r}{a} e^{i\theta} \tag{15}$$

where the terms of $(r/a)^{3/2}$ and higher were dropped. Subsequently, from equation (12) one obtains

$$u_z^P = -\frac{\tau a}{G}\left[1 - \left(\frac{2r}{a}\right)^{1/2}\cos\frac{\theta}{2} + \frac{r}{a}\cos\theta\right] \tag{16}$$

which reduces to (7) at $\theta = \pi$.

Now, in reexamining equation (6) near the tip region, we have

$$u_z^s = \frac{\tau a}{G}\left(1 + \frac{r}{a}\cos\theta\right) \tag{17}$$

The local displacement field which satisfies (3) is then the combination of (16) and (17),

$$u_z = u_z^s + u_z^P = \frac{\tau a}{G}\left(\frac{2r}{a}\right)^{1/2}\cos\frac{\theta}{2} \tag{18}$$

The corresponding stress components of the system are derived from (18),

$$\sigma_{xz} = \frac{\tau(\pi a)^{1/2}}{(2\pi r)^{1/2}}\cos\frac{\theta}{2} \tag{19a}$$

$$\sigma_{yz} = \frac{\tau(\pi a)^{1/2}}{(2\pi r)^{1/2}}\sin\frac{\theta}{2} \tag{19b}$$

It is clear that the system has a stress singularity at the tip of the inhomogeneity similar to the case of a slit crack. It is worth noting that, in the case of a rigid inhomogeneity, the applied shear stress is acting perpendicular to the inhomogeneity segment, whereas in the case of a crack, it is parallel to the crack line.

The above results can also be obtained by the equivalent inclusion method of Eshelby (1957). A rigid line can be considered as a limit case of an elliptic inhomogeneity. Following Yang and Chou (1977), the equivalent eigenstrains for a rigid inhomogeneity (with shear modulus $G_1 \to \infty$) are obtained from the following equations

$$\epsilon_{ij}^c = -\epsilon_{ij}^A \tag{20}$$

where ϵ_{ij}^c are the constrained strains and ϵ_{ij}^A the applied strains.

The constrained strains, ϵ_{xz}^c and ϵ_{yz}^c, can be expressed in terms of the eigenstrains of the equivalent inclusion, ϵ_{xz}^* and ϵ_{yz}^*, i.e.,

$$\epsilon_{xz}^c = \frac{e}{1+e}\epsilon_{xz}^* \tag{21a}$$

$$\epsilon_{yz}^c = \frac{1}{1+e}\epsilon_{yz}^* \tag{21b}$$

where $e = b/a$ is the ratio of the semi-axes, a and b, of the ellipse (Zhang and Zhe, 1981). For a rigid line inhomogeneity $e \to o$ and

$$\epsilon^*_{xz} = -\epsilon^A_{xz}/e = -\sigma^A_{xz}/2Ge \quad (22a)$$

$$\epsilon^*_{yz} = -\epsilon^A_{yz} = -\sigma^A_{yz}/2G \quad (22b)$$

where G is the shear modulus of the inclusion. From equations (22) it is seen that only ϵ^A_{xz} or σ^A_{xz} would induce the stress singularity, in accordance with the results obtained by the complex variable method.

The stress field due to the eigenstrain component ϵ^*_{xz} can be written as (Zhang and Zhe, 1981)

$$\sigma_{xz} = 2\ G\ e\ \epsilon^*_{xz}\left[1 + \frac{1}{R}(x \sin\eta - y\cos\eta)\right] \quad (23a)$$

$$\sigma_{yz} = -2\ G\ e\ \epsilon^*_{xz}\ \frac{1}{R}(x\cos\eta - y\sin\eta) \quad (23b)$$

where

$$R\ e^{i\eta} = \sqrt{a^2 - z^2}, \quad z = x + iy \quad (24)$$

Near the tip of the flat inclusion, equations (23) reduce to

$$\sigma_{xz} = -2\ G\ e\ \epsilon^*_{xz}\left(\frac{\pi a}{2\pi r}\right)^{1/2}\cos\frac{\theta}{2} \quad (25a)$$

$$\sigma_{yz} = -2\ G\ e\ \epsilon^*_{xz}\left(\frac{\pi a}{2\pi r}\right)^{1/2}\sin\frac{\theta}{2} \quad (25b)$$

where r and θ are defined in (14). By substituting (22) into (25) and taking $\sigma^A_{xz} = \tau$, one obtains the sames results as (19).

3 Discussion of the Results

As in the previous paper (Wang et al., 1985), we propose a stress singularity coefficient for the present system,

$$S_{\mathrm{III}} = \lim_{r\to o}\ (2\pi r)^{1/2}\sigma_{xz}(\theta = o)$$
$$= \tau\sqrt{\pi a} \quad (26)$$

This mode of deformation is designated as mode III deformation for a rigid line inhomogeneity. The J integral or the inhomogeneity extension force may also be calculated by using the equivalent inclusion method.

The increase of the elastic energy in the body due to the rigid line inclusion is

$$\Delta W = \pi\ a^2 e\ \sigma^A_{xz}\epsilon^*_{xz}$$
$$= -\frac{\pi\ a^2}{2G}(\sigma^A_{xz})^2 \quad (27)$$

However, the total increase in free energy of the system (the elastic energy in the body and the potential energy of the loading mechanism) is

$$E_{\mathrm{int}} = -\Delta W \quad (28)$$

The J integral is then given by

$$J_{\mathrm{III}} = -\frac{\partial E_{\mathrm{int}}}{\partial\ (2a)} = \frac{\partial\ (\Delta W)}{\partial\ (2a)}$$
$$= -\frac{\pi\ a}{2\ G}(\sigma^A_{xz})^2 = -\frac{\pi a}{2\ G}\tau^2 \quad (29)$$

which is related to S_{III} by

$$J_{\mathrm{III}} = -\frac{1}{2G}S^2_{\mathrm{III}} \quad (30)$$

The negative sign in (29) indicates that the driving force on the rigid line favors a contraction in the length of the segment. This is consistent with the results derived previously for the inplane system (Wang et al., 1985).

To summarize, we have analyzed the stress and displacement fields near the tip of a rigid line inhomogeneity in a two-dimensional elastic system subject to an antiplane shear loading. Similar to the case of the inplane deformation, a square-root stress singularity is identified; the singularity exists only if the shear loading is acting perpendicular to the inhomogeneity segment. In conjunction with the mode I and mode II deformation, a mode III antiplane deformation is defined and the inhomogeneity extensin force is calculated.

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Asymptotic Integration Applied to the Differential Equation for Thin Elastic Toroidal Shells

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The second order differential equation in question is

$$W'' - [i\mu f_1(\phi) + f_2(\phi)]W = \mu F(\phi) \quad (1)$$

where μ is a large parameter and

$$f_1(\phi) = \frac{\sin\phi}{1+\lambda\sin\phi}, \quad f_2(\phi) = \frac{3}{4}\left(\frac{\lambda\cos\phi}{1+\lambda\sin\phi}\right)^2$$
$$F(\phi) = \Omega(\phi)\cos\phi(1+\lambda\sin\phi)^{-1/2} \quad (2)$$

with $\Omega(\phi)$ as an expression for the axial load. A solution of this equation was given by Wei (1944). The homogeneous equation was solved by a method similar to Langers (1931) and the the inhomogeneous equation was solved by the standard formula using an approximation for $F(\phi)$. Clarke (1950) presents a simple particular solution which is widely used, but which is not as accurate as the Langer solution of the homogeneous equation. This particular integral was developed jointly with E. Reissner.

This author presented a more exact solution (Jenssen 1960) which, however, is laborious and has later used the following simplified version:

The solution W is approximated by the function

$$Y = Q(\phi)h(z) + \mu^{1/3}F(o)QT(z) + Y_m \quad (3)$$

where Y_m is the membrane solution

$$Y_m = -i[Q^{-3}F(o) - F(\phi)]\frac{1}{f_1} \quad (4)$$

The functions $h(z)$ and $T(z)$ are solutions of the equations

$$\frac{d^2h}{dz^2} + zh = 0, \quad \frac{d^2T}{dz^2} + zT = -1 \quad (5)$$

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Manuscript received by ASME Applied Mechanics Division, October 28, 1985.