The Method of Fundamental Solutions with Eigenfunctions Expansion Method for 3D Nonhomogeneous Diffusion Equations

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After the successful applications of the combination of the method of fundamental solutions (MFS), the method of particular solutions (MPS), and the eigenfunctions expansion method (EEM) to solve 2D homogeneous and nonhomogeneous diffusion equations by Young et al. (Young et al., Numer Meth Part Differ Equat 22 (2006), 1173), this article intends to extend the same fundamental concepts to calculate more challenging 3D nonhomogeneous diffusion equations. The nonhomogeneous diffusion equations with time-independent source terms and boundary conditions are analyzed by the proposed meshless MFS-MPS-EEM model. Nonhomogeneous diffusion equation in any complex domains can be decomposed into a Poisson equation and a homogeneous diffusion equation by the principle of linear superposition. This approach is proved to be far better off than solutions by using classic method of separation of variables with inefficient multisummation of very sophisticated series expansion from special functions, which can only limit to treat very simple 3D geometries such as cube, cylinder, or sphere. Poisson equation is solved by using the MPS-MFS model, in which the source term in the Poisson equation is first handled by the MPS based on the compactly-supported radial basis functions and the Laplace equation is solved by the MFS. On the other hand, by utilizing the EEM, the homogeneous diffusion equation is first transformed into a Helmholtz equation, which is then solved by the MFS together with the technique of singular value decomposition (SVD) to acquire the eigenvalues and eigenfunctions. After the eigenfunctions are obtained, we can synthesize the diffusion solutions like the orthogonal Fourier series expansions but with only one summation for the series even for multidimensional problems. Numerical results for four case studies of 3D homogeneous and nonhomogeneous diffusion problems show good agreement with the analytical and other numerical solutions, such as finite element method (FEM). Thus, the present numerical scheme has provided a promising meshfree numerical approach to solve 3D nonhomogeneous diffusion equations with time-independent source terms and boundary conditions for very irregular domains. © 2008 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 25: 195–211, 2009

Keywords: 3D nonhomogeneous diffusion equations; eigenfunctions expansion method; method of fundamental solutions; method of particular solutions; singular value decomposition

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I. INTRODUCTION

In recent years, the so-called meshless or meshfree methods have become very attractive numerical alternatives and received a considerable attention as comparing to the classical mesh-dependent numerical methods. The MFS is considered the simplest and most popular method in the category of meshless or meshfree algorithms. The MFS like the boundary element method (BEM) is one of the mesh-reduction methods. One great advantage of the mesh-reduction methods is to reduce one dimension and to save a lot of CPU time and computer memory.

In the applications of the conventional BEM to diffusion problems, for examples, Zhu [1], Zerroukat [2], and Sutradhar et al. [3] all applied the BEM to solve different diffusion equations. Tremendous computational efforts are still required in BEM to calculate the domain integration for the source terms especially in 3D nonhomogeneous diffusion problems. To circumvent those drawbacks, the dual reciprocity method was initiated by Nardini and Brebbia [4] by transforming the domain integral into a boundary integral by a series of radial basis functions. On the other hand, Chen et al. [5] combined the MFS with MPS to deal with the diffusion equations. The combined MFS-MPS model is free from the evaluation of singular integral for solving nonhomogeneous linear operators, as is generally required by the BEM. Therefore the MFS-MPS model, a meshless numerical algorithm, has been considered to be an attractive solver for homogeneous diffusion problems [5–8].

In general, the MFS solutions of homogeneous diffusion equations either use the Laplace transform [5, 6] or the finite difference [7, 8] to discretize the time derivative term. This is due to the fact that the MFS is very capable to solve the spatial domain problems if discretization of transient parts have already been taken care. Chen et al. [6] transformed the diffusion equation into a nonhomogeneous modified Helmholtz equation using the Laplace transform and then used the modified Helmholtz fundamental solution to solve the problems. When the Laplace transform is adopted, the inverse Laplace transform will be needed and sometimes it leads to certain difficulties in retrieving the forward diffusion solutions.

To overcome the drawbacks of the Laplace transform or finite difference discretization in time state, Young et al. [9] used the time-dependent fundamental solutions of diffusion equations directly to solve the multidimensional diffusion equations. Young et al. [10] further extended the time-dependent MFS-MPS model to solve nonhomogeneous diffusion problems. By this time-dependent MFS technique, they were able to solve the nonlinear advection-diffusion-reaction equations if combining with the Eulerian-Lagrangian method (ELM) [11–13]. The only problem in the time-dependent MFS formulation is to lose the advantage of mesh-reduction method since the time dimension is retained. However, there is another way to avoid this disadvantage, which is completely different from the above-mentioned methods to treat the time derivative. This is the so-called EEM in this proposed algorithm. We will use the EEM to remove the time dependence. Yao and Margrave [14] have already used the eigenfunctions transform method to solve the wave equation. As far as time evolution is concerned, the EEM is more feasible and robust for the transient calculations for diffusion and wave equations for any time without using the time marching process. Recently, Young et al. [15] have successfully used this meshless scheme of MFS-MPS-EEM model to solve 2D nonhomogeneous diffusion equations. Both regular and irregular domains are investigated in their numerical tests and the excellent agreement of the numerical results with the analytical solutions indicates the effectiveness of the 2D numerical scheme. Once the numerical model is built, the numerical results at any time can be obtained by a very simple summation procedure from the eigenfunctions like the solutions obtained by the Fourier series expansions. In contrast to the numerical solutions obtained only in discrete time levels by conventional methods, the proposed scheme can provide the continuous results.
along the time axis. In addition, the numerical scheme is free from numerical stability mesh generation, numerical quadrature, and singular integral. In this article, the numerical procedures proposed by Young et al. [15] will be extended to 3D nonhomogeneous diffusion equations with time-independent source terms and boundary conditions.

The MFS-MPS-EEM model with the SVD method is adopted to solve 3D nonhomogeneous diffusion equations in both regular and irregular domains. At first the nonhomogeneous diffusion equation is decomposed into a Poisson equation and a homogeneous diffusion equation without the need of Laplace transform or finite difference or space-time collocation methods to deal with time term as cited earlier. The Poisson equation is analyzed by the well-known MFS-MPS model which is a mesh-free method. Meanwhile, the time-dependent solutions of the homogeneous diffusion equation are directly solved by the superposition principle of the eigenvalues and eigenfunctions obtained by the MFS with SVD. Looking into the physical aspects, it is observed that only few eigenfunctions are needed to represent the solutions for diffusion problems because of the diffusivity character. Moreover, initial condition is used to determine the weighting coefficients of the orthogonal eigenfunctions. The model is applied to 3D diffusion problems with Dirichlet boundary conditions in both regular and irregular domains.

To validate the accuracy of the present method in 3D nonhomogeneous diffusion problems, the analytical and FEM solutions are used. This article is organized as follows. In Section II, the governing equations and the initial as well as boundary conditions (BCs) are elaborated. The numerical discretization of the MFS, MPS, EEM, and SVD schemes are described in details in Section III. In Section IV, we delineate the comparisons of the present results with analytical and FEM computations. The conclusions based on this study and suggestions for further researches are drawn in Section V.

II. GOVERNING EQUATION

Consider a nonhomogeneous diffusion equation with time-independent source term and BCs on computational domain $\Omega$ with the corresponding boundary $\Gamma$:

$$
\frac{\partial u(\tilde{x}, t)}{\partial t} = k \nabla^2 u(\tilde{x}, t) + A(\tilde{x})
$$

in which $\tilde{x} = (x, y, z)$ is the 3D spatial coordinate, $t$ is the time, $k$ is the diffusion coefficient (diffusivity), $A(\tilde{x})$ is the time-independent source function, and $u(\tilde{x}, t)$ is the scalar variable to be determined. The initial condition (IC) of the diffusion equation at initial time $t_0$ is:

$$
u(\tilde{x}, t_0) = B(\tilde{x}) \quad \text{in} \quad \Omega
$$

with the Dirichlet and/or Neumann BCs:

$$
u(\tilde{x}, t) = C(\tilde{x}) \quad \text{on} \quad \Gamma^1
$$

$$
\frac{\partial u}{\partial n}(\tilde{x}, t) = D(\tilde{x}) \quad \text{on} \quad \Gamma^2
$$

where $\Gamma^1 + \Gamma^2$ is equal to the boundary $\Gamma$ and $n$ is the outward normal direction. Moreover, the BC is of the Dirichlet type if only $\Gamma^2 = 0$, of the Neumann type if only $\Gamma^1 = 0$, and of the Robin type if both $\Gamma^1 \neq 0$ and $\Gamma^2 \neq 0$. The boundary conditions $C(\tilde{x})$ and $D(\tilde{x})$ are assumed to be time-independent functions. The augmented data of the problem are $A(\tilde{x})$, $B(\tilde{x})$, $C(\tilde{x})$, and $D(\tilde{x})$, which are all time-independent known functions.
III. NUMERICAL METHOD

Based on the superposition principle of the linear system, the nonhomogeneous diffusion problem with time-independent source terms and BCs can be decomposed into a Poisson equation and a homogeneous diffusion equation. Hence, the solution of the nonhomogeneous diffusion equation will be represented as follows:

$$u(\vec{x}, t) = u_1(\vec{x}) + u_2(\vec{x}, t) \quad \text{in} \quad \Omega$$  (4)

where $u_1(\vec{x})$ satisfies the Poisson equation with nonhomogeneous BCs.

$$\nabla^2 u_1(\vec{x}) = -\frac{1}{k} A(\vec{x})$$

$$u_1(\vec{x}) = C(\vec{x}) \quad \text{on} \quad \Gamma^1$$

$$\frac{\partial}{\partial n} u_1(\vec{x}) = D(\vec{x}) \quad \text{on} \quad \Gamma^2$$  (5)

And $u_2(\vec{x}, t)$ satisfies the homogeneous diffusion equation with homogeneous BCs and inhomogeneous IC.

$$\frac{\partial u_2(\vec{x}, t)}{\partial t} = k \nabla^2 u_2(\vec{x}, t)$$

$$u_2(\vec{x}, t_0) = B(\vec{x}) - u_1(\vec{x}) \quad \text{in} \quad \Omega$$

$$u_2(\vec{x}) = 0 \quad \text{on} \quad \Gamma^1$$

$$\frac{\partial}{\partial n} u_2(\vec{x}) = 0 \quad \text{on} \quad \Gamma^2$$  (6)

In Eq. (5), $u_1(\vec{x})$ is a time-independent function that physically represents steady-state (or quasi-static) solution [9, 15, 16]. Then from the superposition principle we can solve the Poisson equation, Eq. (5), by decomposing the solution into the homogeneous and particular solutions as follows [17]:

$$u_1(\vec{x}) = u_h(\vec{x}) + u_p(\vec{x}),$$  (7)

where $u_h(\vec{x})$ is the homogeneous solution and $u_p(\vec{x})$ is the particular solution.

The particular solution $u_p(\vec{x})$ satisfies the governing equation without BCs.

$$\nabla^2 u_p(\vec{x}) = -\frac{1}{k} A(\vec{x})$$  (8)

The particular solution corresponding to Eq. (8) can be approximated by the MPS for the source term $-(1/k) A(\vec{x})$.

$$-\frac{1}{k} A(\vec{x}) = \sum_{j=1}^{N} a_j^p f (r)$$

$$f (r) = \begin{cases} 
(1 - \frac{r}{\alpha})^2 & r \leq \alpha \\
0 & r > \alpha 
\end{cases}$$  (9)
where $f(r)$ is the compactly-supported radial basis functions (CSRBFs) [18], $\alpha$ is the compact radius, $r = |\vec{x} - \vec{x}_j|$ is the radial distance between the $j$th field point $\vec{x}_j$ and the collocation point $\vec{x}$, and $N$ is the number of collocation nodes. In this study, the collocation points are uniformly distributed in the interior domain as well as on the boundary. After applying Eq. (9) at $N$ collocation points, the unknown coefficient $a_j^p$ can be solved [19]. Therefore, the particular solution $u_p(\vec{x})$ is determined by inverting the Laplace operator of Eq. (8) [20–22]:

$$u_p(\vec{x}) = \sum_{j=1}^{N} \alpha_j^p F(r)$$

$$F(r) = \begin{cases} \frac{r^4}{30\alpha^2} - \frac{r^3}{6\alpha} + \frac{r^2}{6} & r \leq \alpha \\
\frac{a^2}{12} - \frac{a^3}{30r} & r > \alpha \end{cases}$$

where $F(r)$ is the corresponding CSRBFs of $f(r)$ in Eq. (9). On the other hand, the homogeneous solution, $u_h(\vec{x})$, satisfies the Laplace equation as well as the modified BCs:

$$\nabla^2 u_h(\vec{x}) = 0 \quad \text{in } \Omega$$
$$u_h(\vec{x}) = C(\vec{x}) - u_p(\vec{x}) \quad \text{on } \Gamma^1$$
$$\frac{\partial}{\partial n} u_h(\vec{x}) = D(\vec{x}) - \frac{\partial}{\partial n} u_p(\vec{x}) \quad \text{on } \Gamma^2$$

(11)

With the substitution of Eq. (10) into the modified boundary conditions of the homogeneous equation, the results will be a well-posed Laplace problem. Therefore, the MFS is then applied to solve the Laplace equation. The solution of Laplace equation is represented by the MFS form as described by the following:

$$u_h(\vec{x}) = \sum_{j=1}^{M} \alpha_j^h G_1(r)$$

$$G_1(r) = \frac{1}{4\pi r}$$

(12)

where $G_1(r)$ is the fundamental solution of the 3D Laplace equation, which can be obtained by the potential theory. $r = |\vec{x} - \vec{x}_j|$ is the distance between the field point $\vec{x}$ and the $j$th source point $\vec{x}_j$, $M$ is the number of source points. So the solution $u_1(\vec{x})$ of the Poisson equation is obtained by the MFS-MPS model.

After applying the MFS-MPS model for the Poisson equation, the MFS-EEM is then utilized to solve the homogeneous diffusion equation, Eq. (6). We now omit the summation of the infinite series of the eigenfunctions expansion at this moment. Let $u_2(\vec{x}, t) = u_s(\vec{x})e^{-\lambda^2 t}$ and substitute into Eq. (6):

$$\nabla^2 u_s(\vec{x}) + \lambda^2 u_s(\vec{x}) = 0$$
$$u_s(\vec{x}) = 0 \quad \text{on } \Gamma^1$$
$$\frac{\partial}{\partial n} u_s(\vec{x}) = 0 \quad \text{on } \Gamma^2$$

(13)
The homogeneous diffusion equation, Eq. (6), is transformed to the Helmholtz equation, Eq. (13), with homogeneous boundary conditions. Since the homogeneous boundary conditions are encountered, the above eigenvalue problem can be solved by the MFS with the SVD method [23–25]. The solution of the Helmholtz equation by the MFS can be expressed as below:

\[ u_s(\vec{x}) = \sum_{j=1}^{Q} \beta_j G_2(r) \]

where \( G_2(r) \) is the fundamental solution of the 3D Helmholtz equation, which can be obtained by mathematical analysis. \( r = |\vec{x} - \vec{x}_j| \) is the distance between the field point \( \vec{x} \) and the \( j \)th source point \( \vec{x}_j \). \( Q \) is the number of source points. Since Eq. (13) has nontrivial solutions only for some discrete eigenvalues, we use the MFS with SVD [23–25] to obtain the embedding eigenvalues, \( \lambda \), and corresponding eigenfunctions, \( u_s(\vec{x}) \). Then the major advantage of the orthogonal eigenfunctions expansion method is beneficial to this study. The modified IC of the homogeneous diffusion equation is obtained by the following eigenfunction expansion formula:

\[ u_2(\vec{x}, t_0) = B(\vec{x}) - u_1(\vec{x}) = \sum_{j=1}^{\infty} \gamma_j u_{sj}(\vec{x}) e^{-\lambda_j^2 k t_0} \]  

where \( u_{sj}(\vec{x}) \) is the \( j \)th eigenfunctions with corresponding eigenvalue \( \lambda_j \). \( \gamma_j \) is the weighting coefficient of every eigenfunction which is determined by collocating the modified IC of Eq. (15) at some points inside the domain. Depending on physical character of the diffusivity, only the first few eigenfunctions are enough to represent the accurate solutions for the diffusion problems. In our numerical tests, the finite terms are used to replace the infinite series in Eq. (15) and the numerical results will justify the correctness of assumption. Thus we can obtain:

\[ u_2(\vec{x}, t) = \sum_{j=1}^{P} \gamma_j u_{sj}(\vec{x}) e^{-\lambda_j^2 k t} \]  

where \( P \) is the number of adopted eigenfunctions. Therefore, the EEM is capable to obtain time-dependent solution without using the time-marching process, such as Laplace transform or finite difference or space-time collocation methods. From the operational calculus of Eqs. (4)–(16), we can obtain the solutions of Eq. (1) by the linear superposition:

\[ u(\vec{x}, t) = u_1(\vec{x}) + u_2(\vec{x}, t) = u_h(\vec{x}) + u_p(\vec{x}) + u_2(\vec{x}, t) = \sum_{j=1}^{M} \alpha^h_j G_1(r) + \sum_{j=1}^{N} \alpha^p_j F(r) + \sum_{j=1}^{P} \gamma_j u_{sj}(\vec{x}) e^{-\lambda_j^2 k t} \]  

Thus, the solutions of the 3D nonhomogeneous diffusion equations with time-independent source terms and boundary conditions can be obtained by the proposed MFS-MPS-EEM scheme.
is very clear from the observation of Eq. (17) that steady and transient solutions are obtained separately by using present meshless numerical method.

IV. RESULT AND DISCUSSION

Validation for the proposed numerical method is achieved by comparing the results with the analytical and FEM solutions for four 3D diffusion problems with Dirichlet BCs. In addition, the effectiveness of the proposed method is verified by solving 3D homogeneous and nonhomogeneous diffusion problems in regular and irregular domains. In the numerical experiments (Fig. 1), regular domains with analytical solutions for cubic cavity, spherical cavity, and irregular domains without exact solutions for hollow spherical cavity, spiral cavity are considered. The numerical results and comparisons will be shown and discussed in the following section.
Example 1. Consider a cubic cavity of size \([0, a] \times [0, b] \times [0, c]\) [Fig. 1(a)] and the following diffusion equation (DE) with IC and BCs.

\[
\text{DE} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - 6x + 3 \sin x \sin y \sin z = \frac{1}{k} \frac{\partial u}{\partial t}
\]

IC \(u(\vec{x}, 0) = xyz + x^3 + \sin x \sin y \sin z\) in \(\Omega\)

BCs \(u(\vec{x}, t) = x^3 + \sin x \sin y \sin z\) on \(\partial \Gamma\) \(18\)

The analytical solution of the problem is given by

\[
u(\vec{x}, t) = -\frac{8abc}{\pi^3} \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+n+l} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{l\pi z}{c} e^{-\lambda_{mnl}^2 kt} + x^3 + \sin x \sin y \sin z
\]

\[
\lambda_{mnl}^2 = \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + \left( \frac{l\pi}{c} \right)^2 \right).
\]

We set \(a = 0.9, b = 1.1, c = 1.3,\) and \(k = 1.\) Table I shows the numerical results obtained by the MFS with the SVD. The eigenvalue presents the smallest singular value for various \(k,\) which results in a spike when the solution of the Helmholtz equation is numerically singular. The first 8 calculated eigenvalues and eigenfunctions by MFS-EEM with SVD model, using 294 nodes in MFS, and 1000 points to interpolate eigenfunctions are displayed in Fig. 2. These numerically obtained eigenvalues are almost the same as the exact solution of \(\lambda_{mnl} = \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + \left( \frac{l\pi}{c} \right)^2}.\) Comparison shows that the computed and analytical eigenvalues are different only after five significant digits. And the time evolutions of the full field distribution are described in Fig. 3. Except for the results at the beginning, the computed numerical results also show good agreement with analytical solutions at different time stages. The fast decay of field variation also demonstrates the physics underlying the diffusion process clearly. The results generally exhibit good agreement with the analytical solutions at different time stages. Once the position is given, the numerical results can be obtained by a simple summation procedure, Eq. (17). In this example, we also test the maximum absolute error histograms using the present method for different collocating points. And the comparison of the time history of the maximum absolute error for different eigenfunctions is performed when we use 1000 points to interpolate eigenfunctions. As expected more points generally will render better resolution. If the diffusivity is small, physically the first few eigenfunctions are almost qualified to represent the diffusion solutions. The advantage of the method is the capability to obtain the solution for any time by the superposition principle of the first few eigenfunctions, in which their weighting coefficients are dependent on IC.

<table>
<thead>
<tr>
<th>(k)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
</table>

3D NONHOMOGENEOUS DIFFUSION EQUATIONS

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Modal contour (x-y plane) at z=0.65</th>
<th>Modal contour (y-z plane) at x=0.45</th>
<th>Modal contour (x-z plane) at y=0.55</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$ = 5.1167</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_2$ = 6.6107</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_3$ = 7.1169</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_4$ = 7.9205</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_5$ = 8.2566</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_6$ = 8.5382</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_7$ = 8.9585</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_8$ = 9.3383</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

FIG. 2. The first 8 eigenvalues and eigenfunctions for cubic cavity problem for Examples 1 (1000 points for interpolating eigenfunction, 8 eigenfunctions).
FIG. 3. The full-field distribution of $z = 0.65$, $x = 0.45$, and $y = 0.55$ surfaces for Example 1. (a) $t = 0.05$, (b) $t = 0.08$, (c) $t = 0.1$, (d) $t = 0.15$ (1000 points, 8 eigenfunctions). (—: analytical solution; - - - : numerical solution).
Example 2. After simulating the cubic cavity, the proposed numerical scheme is utilized to study the second example in a spherical cavity [Fig. 1(b)]. For simplicity, we choose the nonhomogeneous diffusion equation (DE) with radial and axial symmetry assumptions (independent of $\theta$ and $\phi$) to satisfy the following IC and BCs:

\[
\text{DE} \quad \nabla^2 u(r,t) = \frac{1}{k} \frac{\partial u(r,t)}{\partial t} \\
\text{IC} \quad u(r,t_0) = \frac{0.8r(a^2 - r^2)}{a^2} \quad \text{in } \Omega \\
\text{BCs} \quad u(r,t) = 0 \quad \text{on } \partial \Gamma
\] (20)

The analytical solution of the problem is given by

\[
u(r,t) = \frac{96a}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin \frac{n\pi r}{a} e^{-kn^2\pi^2t/a^2} \quad (21)\]

We set $t_0 = 0$, $a = \text{radius} = 1$ and $k = 1$. The comparison between the present results and the analytical solutions of the time evolutions of the full-field distribution of $z = 0$ surfaces are depicted in Fig. 4, when 6 eigenfunctions and 1062 points to interpolate each eigenfunction are used. Almost identical results are obtained for the numerical and analytical solutions. Except for the results at the beginning, the computed numerical results also show good agreement with analytical solution at different time stages. On the other hand, the further process has demonstrated the maximum absolute error histogram for different numbers of points to interpolate each eigenfunctions, in which more points generally give better results. We have verified that the physical results are generally acceptable for different eigenfunctions.

Example 3. The proposed numerical method is then extended to study a more complex domain problem, the hollow spherical cavity based on the known eigenvalues and eigenmodes [Fig. 1(c)]. We choose a 3D hollow spherical cavity problem with the following IC and BCs:

\[
\text{DE} \quad \frac{\partial u(r,\theta,\phi,t)}{\partial t} = k \nabla^2 u(r,\theta,\phi,t) + f(r,\theta,\phi) \\
\text{IC} \quad u(r,\theta,\phi,t_0) = \frac{\theta(1 - \phi)}{2r} + 3e^{0.2r} + \sin \theta \sin^2 \phi \quad \text{in } \Omega \\
\text{BCs} \quad u(r,\theta,\phi) = 3e^{0.2r} + \sin \theta \sin^2 \phi \quad \text{on } \partial \Gamma
\] (22)

where $f(r,\theta,\phi) = \sin \theta [1 - 2 \cos \phi - 2 \cos 2\phi] - 1.2r^2 \left(0.1 + \frac{1}{r}\right) e^{0.2r}$ is the source term function, and $t_0 = 0$. In this irregular domain, it is rather difficult to get an analytical solution, so, we choose FEM with tetrahedron elements to obtain the numerical results for comparison. The distributions of 19,429 structured tetrahedron elements of FEM meshes and 1150 meshless MFS nodes are applied in this irregular example. Since the eigenvalues and eigenmodes have been obtained from a previous study of solutions of wave or Helmholtz equations [24] up to the first six eigenvalues, we can directly use them to construct the diffusion solutions by the EEM without repetitive works as far as the eigenvalues calculations are concerned. However, we have repeated the works to check the known first six (1–6) eigenvalues and to obtain another remaining four (7–10) eigenvalues and
FIG. 4. The full-field distribution of $z = 0$ surfaces for Example 2. (a) $t = 0.05$, (b) $t = 0.08$, (c) $t = 0.1$, (d) $t = 0.15$ (1062 points, 6 eigenfunctions). (— : analytical solution; --- : numerical solution).

10 eigenmodes. This also demonstrates a big advantage for the proposed MFS-MPS-EEM model to easily combine the wave and diffusion problems together for multiple-dimensional complex domains. Table I shows the first 10 eigenvalues of numerical results of the MFS with the SVD. The eigenvalue presents the smallest singular value for various $k$, which result in a spike when the solution of the Helmholtz equation is numerically singular. Figure 5 portrays the time evolution history of the present numerical code and FEM solutions at four arbitrary locations. It is observed that the present numerical results are almost identical to FEM solutions as the time evolves. In contrast to numerical solutions obtained at some discrete time levels by conventional methods, the results by the MFS-MPS-EEM model always render continuous semi-analytic solutions along the time axis.
Example 4. For the last problem, the proposed method is finally utilized to study the nonhomogeneous diffusion equation in more complicated irregular domain, the spiral cavity [Fig. 1(d)]. A 3D nonhomogeneous diffusion problem with nonhomogeneous source function and IC as well as BCs is given below:

\[
\begin{align*}
\text{DE} & \quad \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + 3e^x - 2\sin y \sin z = \frac{1}{k} \frac{\partial u}{\partial t} \\
\text{IC} & \quad u(\vec{x}, 0) = xyz - 3e^x - \sin y \sin z \quad \text{in } \Omega \\
\text{BCs} & \quad u(\vec{x}, t) = -3e^x - \sin y \sin z \quad \text{on } \partial \Gamma 
\end{align*}
\]

(23)

TABLE II. Comparison of \( u \) with different eigenfunctions for Example 4 \((t = 1\) and 360 points).  

<table>
<thead>
<tr>
<th>Eigenfunctions</th>
<th>Positions 4</th>
<th>Positions 6</th>
<th>Positions 8</th>
<th>FEM solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.3, 0.3, 0.6)</td>
<td>−8.700457</td>
<td>−8.702140</td>
<td>−8.703516</td>
<td>−8.703501</td>
</tr>
<tr>
<td>(0.4, 0.4, 0.8)</td>
<td>−6.364161</td>
<td>−6.365894</td>
<td>−6.366938</td>
<td>−6.366726</td>
</tr>
<tr>
<td>(0.5, 0.5, 1.0)</td>
<td>−5.561102</td>
<td>−5.573606</td>
<td>−5.574195</td>
<td>−5.574472</td>
</tr>
<tr>
<td>(0.6, 0.6, 1.2)</td>
<td>−3.845279</td>
<td>−3.878102</td>
<td>−3.889741</td>
<td>−3.889137</td>
</tr>
<tr>
<td>(0.66, 0.66, 1.33)</td>
<td>−3.513890</td>
<td>−3.526476</td>
<td>−3.526705</td>
<td>−3.527706</td>
</tr>
</tbody>
</table>

In this irregular domain, it is difficult to find an analytical solution; so, we choose FEM with tetrahedron elements to obtain the numerical results for comparison. The distributions of 10,300 structured linear tetrahedron elements for FEM meshes and only 232 meshless MFS nodes are adopted in this example. Table I shows the numerical results of the first 10 eigenvalues with MFS-SVD scheme. The eigenvalue will represent the smallest singular value for various \( k \), which results in a spike when the Helmholtz equation solution is numerically singular. And the first 8 eigenvalues and eigenfunctions are obtained by the MFS-EEM with SVD using 360 points to interpolate eigenfunctions. The numerical solutions at \( t = 1 \) (near steady state solutions) obtained by different eigenfunctions are shown in Table II. It can be observed that the solutions with more eigenfunctions are closer to FEM results. Table III shows the near steady state solutions at \( t = 1 \) of the MFS-MPS-EEM model at some locations for different collocating points for interpolating eigenfunctions. The calculated solutions and the FEM results are very close, which demonstrates the capability of the present model to apply to different shapes of geometry. The results of the MFS-MPS-EEM model with SVD method match very well with the FEM solutions, which use 10,300 linear tetrahedron elements as also depicted in Fig. 6 for some selected points of time evolution as far as transient solutions are concerned. Hence, it is validated that the present numerical method can be used appropriately for problems in irregular domain in a very simple and effective computation.

V. CONCLUSIONS

Usages of the MFS-MPS-EEM model together with the SVD technique to solve the transient inhomogeneous diffusion problems with time-independent source terms and boundary conditions in three dimensions are described in this article. The nonhomogeneous diffusion problems are decomposed into a Poisson equation and a homogeneous diffusion equation. As far as the solution

TABLE III. Comparison of \( u \) with different collocating points for interpolating eigenfunctions for Example 4 \((t = 1 \) and 8 eigenfunctions).  

<table>
<thead>
<tr>
<th>Different collocating points for interpolating eigenfunctions</th>
<th>Positions 150</th>
<th>Positions 250</th>
<th>Positions 360</th>
<th>FEM solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.3, 0.3, 0.6)</td>
<td>−8.037162</td>
<td>−8.604939</td>
<td>−8.703516</td>
<td>−8.703501</td>
</tr>
<tr>
<td>(0.4, 0.4, 0.8)</td>
<td>−6.079654</td>
<td>−6.201250</td>
<td>−6.366938</td>
<td>−6.366726</td>
</tr>
<tr>
<td>(0.5, 0.5, 1.0)</td>
<td>−5.280098</td>
<td>−5.485165</td>
<td>−5.574195</td>
<td>−5.574472</td>
</tr>
<tr>
<td>(0.6, 0.6, 1.2)</td>
<td>−3.440513</td>
<td>−3.814367</td>
<td>−3.889741</td>
<td>−3.889137</td>
</tr>
<tr>
<td>(0.66, 0.66, 1.33)</td>
<td>−3.161461</td>
<td>−3.430545</td>
<td>−3.526705</td>
<td>−3.527706</td>
</tr>
</tbody>
</table>
FIG. 6. Comparison of time evolution of $u$ at (a) $(0.3, 0.3, 0.6)$, (b) $(0.5, 0.5, 1.0)$, and (c) $(0.66, 0.66, 1.33)$ relative location points for Example 4 (360 points, 8 eigenfunctions)
of Poisson equation is concerned, the MFS is adopted to obtain the homogeneous solution and the MPS is utilized to solve the particular solution. On the other hand, the homogeneous diffusion equation is first transformed by the EEM into a Helmholtz equation, which is then solved by the MFS together with SVD scheme to obtain the corresponding eigenvalues and eigenfunctions.

The numerical scheme developed in the present work was validated by comparing with the analytical solutions and FEM results for 3D diffusion problems under Dirichlet boundary conditions. Excellent agreements with the analytical and FEM results indicate the effectiveness of the present method to solve 3D diffusion equations with time-independent source terms and boundary conditions. Hence, it is concluded that the proposed method is capable to obtain reasonable results for multidimensional nonhomogeneous diffusion equations in arbitrary domains if time-independent source terms and boundary conditions are assumed. The extension of the MFS-MPS-EEM model for diffusion problems with time-dependent sources terms will be another interesting topic for future investigation.

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